


# Discrete Structures 2

## Lecture 5:

## Counting Theory

## Part 2

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# Lectures Contents:

- **Counting Theory:**
  - The Basics of Counting
  - The Pigeonhole Principle
  - **Permutations** & **Combinations**
  - **Binomial Coefficients** & **Identities**
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# Introduction

- Many **counting problems** can be solved by:
  - Finding the number of ways to **arrange** a specified number of **distinct** elements of a **set** of a particular size, where the **order** of these **elements** **matters**.
- Many **other** **counting problems** can be solved by:
  - Finding the number of ways to **select** a particular number of **elements** from a **set** of a particular size, where the **order** of the **elements** selected **does not matter**.
- **Examples**:
  - *In how many ways can we select three students from a group of five students to stand in line for a picture?*
  - *How many different committees of three students can be formed from a group of four students?*
- In this lecture, we will present **methods** to answer such questions.

# Permutations: Examples

- **Example 1**:

  1) In how many ways can we select three students from a group of five students to stand in line for a picture?

  2) In how many ways can we arrange all five of these students in a line for a picture?

- **Solution**:

  - First, **note** that the **order** in which we select the students **matters**.
  - There are five ways to select the first student to stand at the start of the line.
  - Once this student has been selected, there are four ways to select the second student in the line.
  - After the first and second students have been selected, there are three ways to select the third student in the line.

# Permutations: Examples

- **Solution** (cont.):
  - By the **product rule**, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.
  - To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way.
  - Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture.
- **Example 1** illustrates how **ordered** arrangements of distinct objects can be counted.
- This leads to some **terminology**.

# Terminology

- A **permutation** of a **set** of distinct objects is an **ordered** arrangement of these objects.
- We also are interested in **ordered** arrangements of **some** of the elements of a **set**.
- An **ordered** arrangement of " ***r*** " elements of a **set** is called an **r-permutation**.

# Permutations: Examples

- **Example 2:**
  - Let **S = { 1, 2, 3 }**
  - The **ordered** *arrangement* **3, 1, 2** is a **permutation** of **S**.
  - The **ordered** *arrangement* **3, 2** is a **2-permutation** of **S**.

- The number of ***r*-permutations** of a **set** with ***n*** elements is denoted by **P(n, r)**
- We can find **P(n, r)** using the **product rule**.

# Permutations: Examples

- **Example 3**:
  - Let **S = { a, b, c }**
  - The ***2*-permutations** of **S** are the **ordered** *arrangements:*
    - (1) **a**, **b** (2) **a**, **c** (3) **b**, **a** (4) **b**, **c** (5) **c**, **a** (6) **c, b**
    - Consequently, there are **six 2-permutations** of this **set** with **three** elements.
  - **There are always six 2-permutations of a set with three elements.**
    - There are three ways to choose the first element of the arrangement.
    - There are two ways to choose the second element of the arrangement:
      - **because** it must be different from the first element.
    - Hence, by the **product rule**, we see that **P(3, 2) = 3 · 2 = 6**. the first element.
  - By the **product rule**, it follows that **P(3, 2) = 3 · 2 = 6**
  - We now use the **product rule** to find a formula for **P(n, *r*)** whenever ***n*** and ***r*** are **positive integers** with **1 ≤ *r* ≤ *n***

# Permutations

- **Theorem 1**:

> If $n$ is a positive integer and $r$ is an integer with $1 \leq r \leq n$, then there are
>
> $$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$
>
> $r$-permutations of a set with $n$ distinct elements.

- **Note** that **P(n, 0) = 1** whenever **n** is a **nonnegative integer**:
  - because there is exactly one way to order zero elements.
  - That is, there is exactly **one list with no elements in it**, namely the **empty list**.
- We now state a useful **corollary** of **Theorem 1**.

# Permutations

- **Corollary 1**:

> If $n$ and $r$ are integers with $0 \leq r \leq n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

- By **Theorem 1**, we know that if ***n*** is a **positive integer**, then **P(n, n) = n!**
  - We will illustrate this result with some examples.

# Permutations: Examples

- **Example 4**:
  - How many ways are there to select a **first**-prize winner, a **second**-prize winner, and a **third**-prize winner from **100** different people who have entered a contest?
- **Solution**:
  - Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of **100** elements, that is:
    - the number of ***3*-permutations** of a set of **100** elements.
  - Consequently, the **answer** is **P(100, *3*) = 100 · 99 · 98 = 970,200**

# Permutations: Examples

- **Example 5**:
  - Suppose that there are eight runners in a race.
  - The winner receives a **gold** medal, the **second-place** finisher receives a **silver** medal, and the **third-place** finisher receives a **bronze** medal.
  - How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?
- **Solution**:
  - The number of different ways to award the medals is the number of ***3*-permutations** of a set with eight elements.
  - Hence, there are $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.

# Permutations: Examples

- **Example 6**:
  - Suppose that a saleswoman has to visit eight different cities.
  - She must begin her trip in **a specified city**, but she can visit the other seven cities in **any order** she wishes.
  - How many possible orders can the saleswoman use when visiting these cities?
- **Solution**:
  - The number of possible **paths** between the cities is the **number of permutations of seven elements**, because the **first city is determined**, but the **remaining seven can be ordered arbitrarily**.
  - Consequently, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour.
  - If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path.
    - She must consider a total of **5040** paths!

# Permutations: Examples

- **Example 7**:
  - How many permutations of the letters ABCDEFGH contain the string **ABC** ?
- **Solution**:
  - Because the letters **ABC** must occur as a **block**, we can find the answer by finding the number of permutations of **six** objects, namely, the **block ABC** and the individual letters D, E, F, G, and H.
  - Because these **six** objects can occur in any order, there are **6! = 720** permutations of the letters ABCDEFGH in which **ABC** occurs as a **block**.

# Combinations

- We now turn our attention to counting **unordered** selections of objects.
- We begin by solving a question posed in the introduction to this section of the lecture.
- **Example 1**:
  - How many different committees of three students can be formed from a group of four students?
- **Solution**:
  - To answer this question, we need only to find the number of subsets with three elements from the set containing the four students.
  - We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group.
  - This means that there are four ways to choose the three students for the committee, where the **order** in which these students are chosen **does not matter**.

# Combinations

- **Example 1** illustrates that many counting problems can be solved by:
  - Finding the number of **subsets** of a particular **size** of a set with ***n*** elements, where ***n*** is a **positive integer**.
- An ***r*-combination** of elements of a **set** is an **unordered selection** of ***r*** elements from the **set**.
- Thus, an ***r*-combination** is simply "a **subset of the set with *r* elements**"

# Combinations: Examples

- **Example 2**:
  - Let **S** be the set **{ 1, 2, 3, 4 }**
  - Then **{1, 3, 4}** is a **3-combination** from **S**.
    - **Note** that **{ 4, 1, 3 }** is the same **3-combination** as **{1, 3, 4}**
      - because the **order** in which the elements of a set are listed **does not matter**.

- The number of ***r*-combinations** of a **set** with ***n*** distinct elements is denoted by **C(n, *r*)**
  - **Note** that **C(n, r)** is also denoted by $\binom{n}{r}$ and is called a **binomial coefficient**.
  - We will learn where this terminology comes from in the coming slides.

# Combinations: Examples

- **Example 3**:
  - We see that **C(4, 2) = 6**, because the **2-combinations** of **{a, b, c, d}** are the **six subsets** **{a, b}**, **{a, c}**, **{a, d}**, **{b, c}**, **{b, d}**, and **{c, d}**

- We can determine the number of ***r*-combinations** of a **set** with ***n*** elements using the formula for the number of ***r*-permutations** of a **set**.
- To do this, note that the ***r*-permutations** of a **set** can be obtained by first forming ***r*-combinations** and then **ordering the elements in these combinations**.

# Combinations

- **Theorem 2**:

> The number of $r$-combinations of a set with $n$ elements, where $n$ is a nonnegative integer and $r$ is an integer with $0 \leq r \leq n$, equals
>
> $$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

- **Example 4**:
  - How many poker hands of five cards can be dealt from a standard deck of **52** cards?
  - Also, how many ways are there to select **47** cards from a standard deck of **52** cards?

# Combinations: Examples

- **Solution**:
  - Because the **order** in which the five cards are dealt from a deck of 52 cards does not matter, there are

$$C(52, 5) = \frac{52!}{5!47!}$$

different hands of five cards that can be dealt.

  - To compute the value of **C**(**52**, **5**), first divide the numerator and denominator by 47! to obtain

$$C(52, 5) = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

  - We find that **C(52, 5) = 2,598,960**
  - Consequently, there are **2,598,960** different poker hands of five cards that can be dealt from a standard deck of 52 cards.

# Combinations: Example 11 (cont.)

- Note that there are

$$C(52, 47) = \frac{52!}{47!5!}$$

different ways to select **47** cards from a standard deck of **52** cards.

- We **do not need** to compute this value because **C(52, 47) = C(52, 5)**
  - Only the order of the factors **5!** and **47!** is different in the *denominators* in the formulae for these quantities.
- It follows that there are also **2,598,960** different ways to select **47** cards from a standard deck of **52** cards.

# Combinations

- In **Example 4**, we observed that **C(52, 5) = C(52, 47)**.
- The **identity** **C(52, 5) = C(52, 47)** is a **special case** of the useful identity for the number of ***r*-combinations** of a set given in **Corollary 2**.
- **Corollary 2**:

> Let $n$ and $r$ be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

# Combinations: Examples

- **Example 5**:
  - How many ways are there to select **five** players from a **10**-member tennis team to make a trip to a match at another school?
- **Solution**:
  - The answer is given by the number of **5-combinations** of a set with **10** elements.
  - By **Theorem 2**, the number of such combinations is

$$C(10, 5) = \frac{10!}{5!\,5!} = 252$$

# Combinations: Examples

- **Example 6**:
  - A group of **30** people have been trained as astronauts to go on the first mission to Mars.
  - How many ways are there to select a crew of **six** people to go on this mission (assuming that all crew members have the same job)?
- **Solution**:
  - The number of ways to select a crew of **six** from the pool of **30** people is the number of **6-combinations** of a set with **30** elements, because the order in which these people are chosen **does not matter**.
  - By **Theorem 2**, the number of such combinations is

$$C(30, 6) = \frac{30!}{6!\,24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593{,}775$$

# Combinations: Examples

- **Example 7**:
  - How many bit strings of length ***n*** contain exactly ***r*** **1s**?
- **Solution**:
  - The **positions** of ***r*** **1**s in a bit string of length ***n*** form an ***r*-combination** of the set **{1**, **2**, **3**,…, ***n*}**.
  - Hence, there are **C(*n*, *r*)** bit strings of length ***n*** that contain exactly ***r*** **1**s.

# Combinations: Examples

- **Example 8**:
  - Suppose that there are **9** faculty members in the **mathematics** department and **11** in the **computer science** department.
  - How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?
- **Solution**:
  - By the **product rule**, the answer is the product of the number of **3-combinations** of a set with **nine** elements and the number of **4-combinations** of a set with **11** elements.
  - By **Theorem 2**, the number of ways to select the committee is

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720$$

# Binomial Coefficients and Identities: Introduction

- As we remarked in **Slide 18**, the number of ***r*-combinations** from a **set** with ***n*** elements is often denoted by $\binom{n}{r}$
- This number is also called a **binomial coefficient**
  - **Because** these numbers occur as **coefficients** in the expansion of powers of **binomial** expressions such as **$(a + b)^n$**
- We will discuss the binomial theorem, which gives a power of a binomial expression as a **sum of terms** involving **binomial coefficients**.

# The Binomial Theorem

- The **binomial theorem** gives the **coefficients** of the expansion of powers of **binomial** *expressions*.
- A **binomial** expression is simply the sum of two terms, such as:
  - **x + y**
- The **terms** can be products of constants and variables, but that does not concern us here.
- **Example 1** illustrates how the **coefficients** in a typical expansion can be found and prepares us for the statement of the **binomial theorem**.

# Binomial Coefficients: Example 1

- The expansion of $(x + y)^3$ can be found using **combinatorial reasoning** instead of multiplying the three terms out.
- When $(x + y)^3 = (x + y)(x + y)(x + y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added.
  - Terms of the form $x^3$, $x^2y$, $xy^2$, and $y^3$ arise.
- To obtain a term of the form $x^3$, an $x$ must be chosen in each of the sums, and this can be done in only one way.
  - Thus, the $x^3$ term in the **product** has a **coefficient** of **1**

# Binomial Coefficients: Example 1 (cont.)

- To obtain a term of the form $\mathbf{x^2y}$, an **x** must be chosen in **two** of the three sums (and consequently a **y** in the other sum).
  - Hence, the number of such terms is the number of **2-combinations** of three objects, namely, $\binom{3}{2}$
- Similarly, the number of terms of the form $\mathbf{xy^2}$ is the number of ways to pick one of the three sums to obtain an x (and consequently take a y from each of the other two sums).
  - This can be done in $\binom{3}{1}$ ways.
- Finally, the only way to obtain a $\mathbf{y^3}$ term is to choose the **y** for each of the three sums in the product, and this can be done in exactly one way.
- Consequently, it follows that

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) = (xx+xy+yx+yy)(x+y)\\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

# The Binomial Theorem

- **Theorem 1**:

**THE BINOMIAL THEOREM** Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- Some computational uses of the **binomial theorem** are illustrated in **Examples 1–3**.

# The Binomial Theorem: Examples

- **Example 1**:
  - What is the expansion of **$(x + y)^4$** ?
- **Solution:**
  - From the **binomial theorem** it follows that

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4.$$

# The Binomial Theorem: Examples

- **Example 2**:
  - What is the **coefficient** of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$ ?
- **Solution:**
  - From the **binomial theorem** it follows that this **coefficient** is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

# The Binomial Theorem: Examples

- **Example 3**:
  - What is the coefficient of $\mathbf{x^{12}y^{13}}$ in the expansion of $\mathbf{(2x - 3y)^{25}}$?
- **Solution:**
  - First, **note** that this expression equals $\mathbf{(2x + (-3y))^{25}}$
  - By the **binomial theorem**, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}$$

# The Binomial Theorem: **Example 4** (cont.)

- Consequently, the **coefficient** of $\mathbf{x^{12}y^{13}}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}$$

- **Note** that another way to find the **solution** is to first use the **binomial theorem** to see that

$$(u+v)^{25} = \sum_{j=0}^{25} \binom{25}{j} u^{25-j} v^j$$

- Setting **u = 2x** and **v = −3y** in this equation yields the same result.

# Binomial Identities

- We can prove some useful **identities** using the **binomial** theorem, as **Corollaries 1**, **2**, and **3** demonstrate.

- **Corollary 1**:

Let $n$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

- **Corollary 2**:

Let $n$ be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

# Binomial Identities

- **Corollary 3**:

Let $n$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

# Pascal's Identity and Triangle

- The **binomial coefficients** satisfy many different identities.
- We introduce one of the most important of these now.

- **Theorem 2:**

**PASCAL'S IDENTITY** Let $n$ and $k$ be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

# Pascal's Identity and Triangle

- **Remark: Pascal's identity**, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers **n**, can be used to recursively define binomial coefficients.
- This **recursive** definition is useful in the computation of **binomial coefficients** because only addition, and not multiplication, of integers is needed to use this recursive definition.
- **Pascal's identity** is the basis for a geometric arrangement of the **binomial coefficients** in a **triangle**, as shown in **Figure 1**.
- The ***n*th row** in the **triangle** consists of the **binomial coefficients**

$$\binom{n}{k}, \; k = 0, 1, \ldots, n.$$

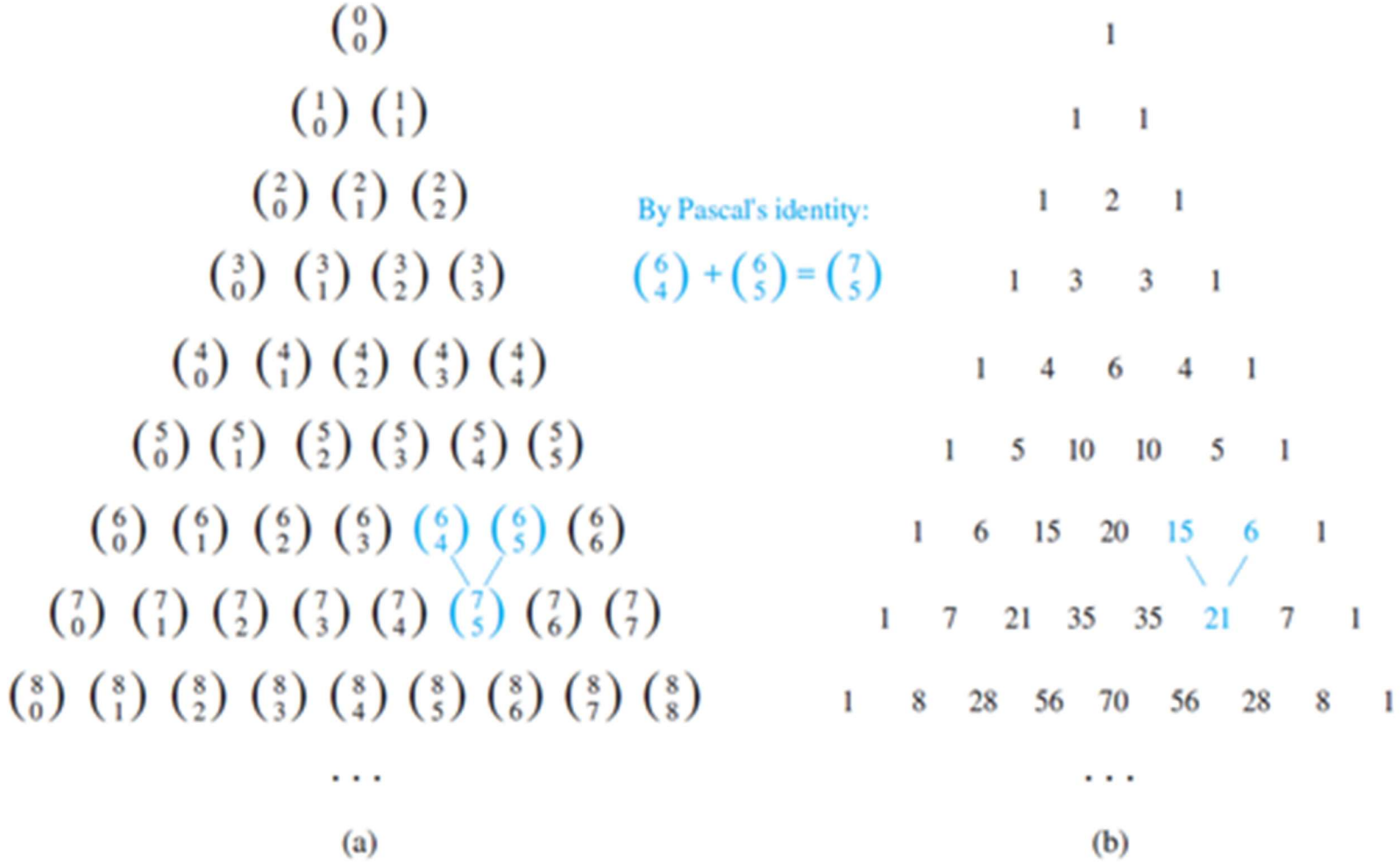



**FIGURE 1** **Pascal's triangle.**

# Any Questions?

- Have a look at the "**Extra Examples**" on the textbook's website.
- Keep reviewing this lecture *regularly, as well as other lectures*.
- Have a look at the **exercises** at the end of the sections we covered in this lecture in **Chapter 6** and do your best with those ones similar to what we presented in this lecture.
  - You should get more *examples* and *exercises* in the *tutorials*.
- Ask *me or ask your peers* if you have any questions.
  - Please, do not wait until the end of the semester or even right before exams.

All the best!