

Discrete Structures 2

Lecture 6:

Relations

Part 1

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Lecture's content is taken from "Discrete Mathematics and Its Applications" 8th edition by Kenneth H. Rosen. McGraw-Hill, 2019.

Lectures Contents:

❖ Relations:

- **Relations and Their Properties**
- *n*-ary Relations and Their Applications
- Representing Relations
- Closures of Relations
- Equivalence Relations
- Partial Orderings

Lecture Contents

- **Relations and Their Properties:**
  - Cartesian Product
  - Relations:
    - Binary Relation
    - Representing Binary Relations
  - Functions as Relations
  - Relations on a Set
  - Properties of Relations:
    - Reflexive & Irreflexive
    - Symmetric, Antisymmetric & Asymmetric
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Cartesian Product (*Review*)

- Let **A** = {a_1, a_2, …, a_k} and **B** = {b_1, b_2, …, b_m}
- The **Cartesian Product A** x **B** is defined by a set of pairs { (a_1, b_1), (a_1, b_2), …, (a_1, b_m), …, (a_k, b_m) }
- **Cartesian Product** defines a product set, or a set of all **ordered** arrangements of elements in sets in the **Cartesian Product**.

Relations

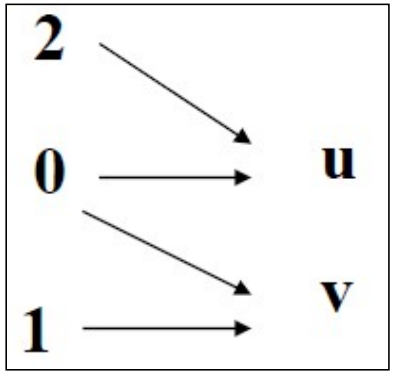
- If we want to describe a relationship between elements of two sets **A** and **B**, we can use **ordered pairs** with their first element taken from **A** and their second element taken from **B**.
  - Since this is a relation between **two sets**, it is called a **binary relation**.
- **Definition:** Let **A** and **B** be sets. A **binary relation** from **A** to **B** is a *subset* of $A \times B$
  - In other words, for a **binary relation** **R** we have $R \subseteq A \times B$
- We use the notation **a R b** to denote that $(a, b) \in R$ and **a** $\not R$ **b** to denote that $(a, b) \notin R$

Binary Relation

- **Example:**
  - Let **A** = { a, b, c } and **B** = { 1, 2, 3 }
  - Is **R** = { (a, 1), (b, 2), (c, 2) } a relation from **A** to **B** ? **Yes**
  - Is **Q** = { (1, a), (2, b) } a relation from **A** to **B**? **No**
  - Is **P** = { (a, a),(b, c),(b, a) } a relation from **A** to **A**? **Yes**

Representing Binary Relations

- We can **graphically** represent a **binary relation R** as follows:
  - If **a R b** then draw an arrow from **a** to **b**
    - **a** $\rightarrow$ **b**



- **Example:**
  - Let **A** = { 0, 1, 2 }, **B** = { u, v } and **R** = { (0, u), (0, v), (1, v), (2, u) }
    - Note: **R** $\subseteq$ **A** x **B**
    - **Graph**
      - See the figure at right.

Representing Binary Relations

- We can represent a **binary relation R** by a **table** showing (marking) the ordered pairs of **R**.
- **Example:**
  - Let **A** = { 0, 1, 2 }, **B** = { u, v } and **R** = { (0, u), (0, v), (1, v), (2, u) }

R	u	v
0	x	x
1		x
2	x	

OR

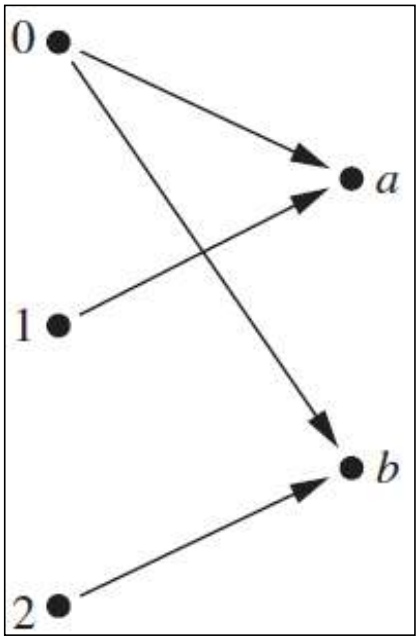
R	u	v
0	1	1
1	0	1
2	1	0

Relations

- When **(a, b)** belongs to **R**, "**a**" is said to be **related** to "**b**" by **R**.
- **Example:** Let **P** be a set of people, **C** be a set of cars, and **D** be the *relation* describing which person drives which car(s).
  - **P** = { Carl, Suzanne, Peter, Carla }
  - **C** = { Mercedes, BMW, tricycle}
  - **D** = { (Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle) }
  - This **means** that:
    - Carl drives a Mercedes,
    - Suzanne drives a Mercedes and a BMW,
    - Peter drives a tricycle, and
    - Carla does not drive any of these vehicles.

Functions as Relations

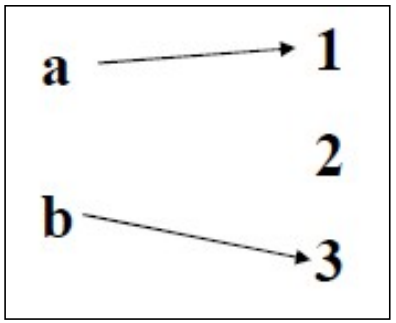
- You might remember that a **function f** from a **set A** to a **set B** *assigns a unique element* of **B** to *each element of* **A**
- The **graph** of **f** is the set of ordered pairs (**a**, **b**) such that **b** = **f**(**a**)
- Since the **graph** of **f** is a subset of **A** × **B**, it is a **relation** from **A** to **B**
- Moreover, for each element **a** of **A**, there is exactly one **ordered pair** in the **graph** that has **a** as its first element.
- However, **relations** represent **one-to-many relationships** between elements in **A** and **B**
- **Example:**
  - See the figure at the right.



Functions as Relations

- Conversely, if R is a *relation* from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.
- This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$
- What is the difference between a relation and a function from A to B?
  - A function defined on sets A and B ($A \rightarrow B$) assigns to each element in the domain set A exactly one element from B.
  - So, a *function* is a **special relation**.

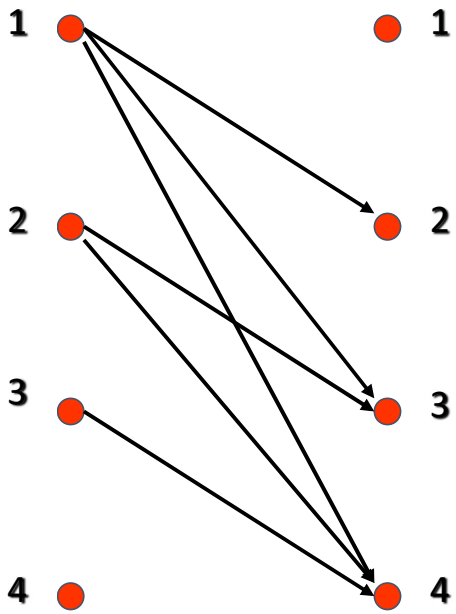
- **Relations** are more general than functions. A **function** is a relation where exactly one element of ***B*** is related to each element of ***A***.



Relations on a **Set**

- **Definition:** A *relation* on the set **A** is a *relation* from **A** to **A**
  - In other words, a *relation* on the set **A** is a subset of **A**×**A**
- **Example:**
  - Let **A** = {1, 2, 3, 4}
  - Which ordered pairs are in the *relation* **R** = { (a, b) | a < b } ?
- **Solution:**
  - **R** = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Relations on a **Set**



R	1	2	3	4
1		x	x	x
2			x	x
3				x
4				

Relations on a Set

- **Example**:
  - Let **A** = { 1, 2, 3, 4 } and R_{div} = { (a, b) | a divides b}
  - What does R_{div} consist of ?
- **Solution**:
  - R_{div} = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) }

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relations on a Set

- **Example**:
  - Let **A** = { 1, 2, 3, 4 }
  - Define a $R_{\neq}$ b if and only if a ≠ b
- **Solution**:
  - $R_{\neq}$ = { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) }

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Relations on a **Set**

- How many different relations can we define on a set **A** with *n* elements?
- A relation on a set **A** is a subset of **A**×**A**
  - How many elements are in **A**×**A** ?
- There are $\mathbf{n^2}$ elements in **A**×**A**, so how many subsets (= relations on **A**) does **A**×**A** have? Where |**A**|= **n**
- The number of subsets that we can form out of a set with ***m*** elements is $\mathbf{2^m}$
  - Therefore, $\mathbf{2^{n^2}}$ subsets can be formed out of **A**×**A**
    - $\mathbf{2}^{|AxA|} = \mathbf{2^{n^2}}$
- **Answer:** We can define $\mathbf{2^{n^2}}$ different relations on **A**

Properties of Relations: Reflexive

- We will now look at some useful ways to classify relations.
- **Definition:** A relation **R** on a set **A** is called **reflexive** if $(a, a) \in \mathbf{R}$ for every element $a \in \mathbf{A}$
- **Example 1:** Are the following relations on {1, 2, 3, 4} **reflexive**?

  R = { (1, 1), (1, 2), (2, 3), (3, 3), (4, 4) } **No**

  R = { (1, 1), (2, 2), (2, 3), (3, 3), (4, 4) } **Yes**

  R = { (1, 1), (2, 2), (3, 3) } **No**

- If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

- **Definition:** A relation on a set **A** is called **irreflexive** if $(a, a) \notin \mathbf{R}$ for every element $a \in \mathbf{A}$
  - See **Example 3**

Properties of Relations: Reflexive

- **Example 2:**
  - Assume relation $\mathbf{R_{div}}$ = { (a, b), if a | b } on **A** = { 1, 2, 3, 4 }
  - Is $\mathbf{R_{div}}$ **reflexive**?
- **Solution:**
  - $\mathbf{R_{div}}$ = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) }
  - **Answer**: **Yes**
    - Because (1, 1), (2, 2), (3, 3), and (4, 4) $\in$ $\mathbf{R_{div}}$

Properties of Relations: Reflexive

- A relation **R** is **reflexive** if and only if **MR** has **1** in every position on its **main diagonal**.
- **Example:**
  - $\mathbf{R_{div}}$ = { (a b), if a | b } on **A** = { 1, 2, 3, 4 }
  - $\mathbf{R_{div}}$ = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) }

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Properties of Relations: Irreflexive

- **Example 3:**
  - Assume relation $\mathbf{R}_{\neq}$ on **A** = {1, 2, 3, 4}, such that **a** $\mathbf{R}_{\neq}$ **b** if and only if **a** ≠ **b**
  - Is $\mathbf{R}_{\neq}$ irreflexive?
- **Solution:**
  - $\mathbf{R}_{\neq}$ = { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) }
  - **Answer**: Yes
    - Because (1, 1), (2, 2), (3, 3) and (4, 4) ∉ $\mathbf{R}_{\neq}$

Properties of Relations: Irreflexive

- A relation R is irreflexive if and only if **M**R has **0** in every position on its **main diagonal**.
- **Example:**
  - $\mathbf{R}_{\neq}$ = { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) }

$$M\mathbf{R}_{\neq} = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Properties of Relations: Symmetric, Antisymmetric & Asymmetric

- **Definitions:**
  - A relation **R** on a set **A** is called **symmetric** if $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}$ whenever $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ (i.e., $\forall \mathbf{a},\mathbf{b} \in \mathbf{A}\ (\mathbf{a}, \mathbf{b}) \in \mathbf{R} \rightarrow (\mathbf{b}, \mathbf{a}) \in \mathbf{R}$)
  - A relation **R** on a set **A** is called **antisymmetric** if $\mathbf{a} = \mathbf{b}$ whenever $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$ and $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}$, i.e., $[(\mathbf{a}, \mathbf{b}) \in \mathbf{R} \wedge (\mathbf{b}, \mathbf{a}) \in \mathbf{R}] \rightarrow \mathbf{a} = \mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in \mathbf{A}$
  - A relation **R** on a set **A** is called **asymmetric** if $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$ implies that $(\mathbf{b}, \mathbf{a}) \notin \mathbf{R}$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{A}$

Properties of Relations: Symmetric, Antisymmetric & Asymmetric

- **Example:**
  - Are the following relations on {1, 2, 3, 4} Symmetric, Antisymmetric, or Asymmetric?

R = { (1, 1), (1, 2), (2, 1), (3, 3), (4, 4) } **Symmetric**

R = { (1, 1) } **Symmetric** & **Antisymmetric**

R = { (1, 3), (3, 2), (2, 1) } **Antisymmetric** & **Asymmetric**

R = { (4, 4), (3, 3), (1, 4) } **Antisymmetric**

Properties of Relations: Symmetric

- **Example 1**:
  - Assume relation $\mathbf{R_{div}}$ = { (a b), if a | b } on **A** = { 1, 2, 3, 4 }
  - Is $\mathbf{R_{div}}$ **Symmetric**?
- **Solution**:
  - $\mathbf{R_{div}}$ = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) }
  - **Answer**: **No**
    - Because (1, 2) $\in$ $\mathbf{R_{div}}$ but (2, 1) $\notin$ $\mathbf{R_{div}}$

Properties of Relations: Symmetric

- **Example 2**:
  - Assume relation $\mathbf{R}_{\neq}$ on **A** = {1, 2, 3, 4}, such that **a** $\mathbf{R}_{\neq}$ **b** if and only if **a** ≠ **b**
  - Is $\mathbf{R}_{\neq}$ **Symmetric**?
- **Solution:**
  - $\mathbf{R}_{\neq}$ = { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) }
  - **Answer**: Yes
    - Because If (**a**, **b**) $\in$ $\mathbf{R}_{\neq}$ $\rightarrow$ (**b**, **a**) $\in$ $\mathbf{R}_{\neq}$

Properties of Relations: Symmetric

- A relation **R** is **symmetric** if and only if $m_{ij} = m_{ji}$ for all *i,j*
- **Example:**
  - $\mathbf{R}_{\neq}$ = { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) }

$$M\mathbf{R}_{\neq} = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Properties of Relations: Antisymmetric

- **Example:**
  - Relation $\mathbf{R_{fun}}$ on **A** = {1, 2, 3, 4} defined as:
    - $\mathbf{R_{fun}}$ = { (1, 2), (2, 2), (3, 3) }
  - Is $\mathbf{R_{fun}}$ **antisymmetric** ?
- **Solution:**
  - **Yes**. It is **antisymmetric**

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $\mathbf{m_{ij} = 1 \rightarrow m_{ji} = 0}$ for $\mathbf{i \neq j}$

Properties of Relations: Transitive

- **Definition:** A relation **R** on a set **A** is called **transitive** if whenever (**a**, **b**)$\in$**R** and (**b**, **c**)$\in$**R**, then (**a**, **c**)$\in$**R** for **a**, **b**, **c**$\in$**A**
  - i.e., [(**a**, **b**) $\in$ **R** $\wedge$ (**b**, **c**) $\in$ **R**] $\rightarrow$ (**a**, **c**) $\in$ **R** for all **a**, **b**, **c** $\in$ **A**
- **Example:**
  - Are the following relations on {1, 2, 3, 4} **transitive**?

R = { (1, 1), (1, 2), (2, 2), (2, 1), (3, 3) } **Yes**

R = { (1, 3), (3, 2), (2, 1) } **No**

R = { (2, 4), (4, 3), (2, 3), (4, 1) } **No**

Properties of Relations: Transitive

- **Example 1:**
  - Assume relation $\mathbf{R_{div}}$ = { (a, b), if a | b } on **A** = { 1, 2, 3, 4 }
  - Is $\mathbf{R_{div}}$ **Transitive**?
- **Solution:**
  - $\mathbf{R_{div}}$ = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) }
  - **Answer**: **Yes**

Properties of Relations: Transitive

- **Example 2:**
  - Assume relation $\mathbf{R}_{\neq}$ on **A** = {1, 2, 3, 4}, such that **a** $\mathbf{R}_{\neq}$ **b** if and only if **a ≠ b**
  - Is $\mathbf{R}_{\neq}$ **Transitive**?
- **Solution:**
  - $\mathbf{R}_{\neq}$ = { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) }
  - **Answer**: **No**
    - It is **not transitive** since (1, 2) ∈ R and (2, 1) ∈ R but (1, 1) is **not** an element of $\mathbf{R}_{\neq}$

Properties of Relations: Transitive

- **Example 3:**
  - Relation $\mathbf{R_{fun}}$ on **A** = {1, 2, 3, 4} defined as:
    - $\mathbf{R_{fun}}$ = { (1, 2), (2, 2), (3, 3) }
  - Is $\mathbf{R_{fun}}$ **Transitive** ?
- **Solution:**
  - **Yes**. It is **Transitive**

Counting Relations

- **Example:** How many different **reflexive** relations can be defined on a **set A** containing ***n*** elements?
- **Solution:** Relations on **R** are subsets of **A**×**A**, which contains $\mathbf{n^2}$ elements.
- Therefore, **different** relations on **A** can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.
- A **reflexive** relation, however, **must** contain the ***n*** elements (**a**, **a**) for every $\mathbf{a} \in \mathbf{A}$
- Consequently, we can only choose among $\mathbf{n^2 - n = n(n-1)}$ elements to generate **reflexive** relations, so there are $\mathbf{2^{n(n-1)}}$ of them.

Combining Relations

- **Relations** are **sets**, and therefore, we can apply the usual **set operations** to them.
- If we have two relations R_1 and R_2, and both of them are from a set **A** to a set **B**, then we can combine them to:
  - $R_1 \cup R_2$
  - $R_1 \cap R_2$
  - $R_1 - R_2$
  - $R_2 - R_1$
- In each case, the **result** will be **another relation from A to B**

Combining Relations

- … and there is **another important way to combine relations**.
- **Definition:** Let **R** be a relation from a set **A** to a set **B** and **S** a relation from **B** to a set **C**. The **composite** of **R** and **S** is the relation consisting of ordered pairs (**a**, **c**), where **a**$\in$**A**, **c**$\in$**C**, and for which there exists an element **b**$\in$**B** such that (**a**, **b**)$\in$**R** and (**b**, **c**)$\in$**S**. We denote the composite of **R** and **S** by **S** $\circ$ **R**
- In other words, if relation **R** contains a pair (**a**, **b**) and relation **S** contains a pair (**b**, **c**), then **S** $\circ$ **R** contains a pair (**a**, **c**)

Combining Relations

- **Example:** Let **D** and **S** be relations on **A** = {1, 2, 3, 4}
- **D** = { (a, b) | b = 5 - a } "b equals (5 – a)"
- **S** = { (a, b) | a < b } "a is smaller than b"
- **D** = { (1, 4), (2, 3), (3, 2), (4, 1) }
- **S** = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }
- **S** ∘ **D** = { (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4) }
- **D** maps an element a to the element (5 – a), and afterwards **S** maps (5 – a) to all elements larger than (5 – a), resulting in **S** ∘ **D** = **{ (a, b) | b > 5 – a }** or **S** ∘ **D** = **{ (a, b) | a + b > 5 }**

Combining Relations

- We already know that **functions** are just **special cases** of **relations** (namely those that *map each element in the **domain** onto exactly one element in the **codomain***).
- If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these **relations** will be exactly the same as the composite of the **functions** (as defined earlier).

Combining Relations

- **Definition:** Let **R** be a relation on the set **A**. The powers $\mathbf{R}^n$, n = 1, 2, 3, …, are defined inductively by
  - $\mathbf{R}^1 = \mathbf{R}$
  - $\mathbf{R}^{n+1} = \mathbf{R}^n \circ \mathbf{R}$

- In other words:
- $\mathbf{R}^n = \mathbf{R} \circ \mathbf{R} \circ \ldots \circ \mathbf{R}$ (n times the letter **R**)

Combining Relations

- **Theorem:** The relation **R** on a set **A** is **transitive** if and only if $R^n \subseteq R$ for all positive integers **n**
- Remember the definition of **transitivity**:
- **Definition:** A relation **R** on a set **A** is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$
- The **composite** of **R** with **itself** contains exactly these **pairs** (a, c)
- Therefore, for a **transitive** relation **R**, $R \circ R$ **does not** contain any **pairs** that are **not** in **R**, so $R \circ R \subseteq R$
- Since $R \circ R$ does not introduce any pairs that are not already in **R**, it must also be **true** that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$

Any Questions?

- Have a look at the "**Extra Examples**" on the textbook's website.
- Keep reviewing this lecture *regularly, as well as other lectures*.
- Have a look at the **exercises** at the end of the sections we covered in this lecture in **Chapter 9** and do your best with those ones similar to what we presented in this lecture.
  - You should get more *examples* and *exercises* in the *tutorials*.
- Ask *me or ask your peers* if you have any questions.
  - Please, do not wait until the end of the semester or even right before exams.

All the best!