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Discrete Structures 2

Graphs

Part 1

Computer Science & Artificial Intelligence Department
Umm Al-Qura University



Acknowledgment: Slides are mainly adopted from Rosen's book supplements.

AGENDA

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- Graphs and Graph Models
- Graph Terminology and Special Types of Graphs:
  - Basic Graph Terminology
  - Some Special Types of Graphs
  - Bipartite Graphs
  - New Graphs from Old
- Representing Graphs and Graph Isomorphism:
  - Adjacency Lists
  - Adjacency Matrices
  - Isomorphism of Graphs



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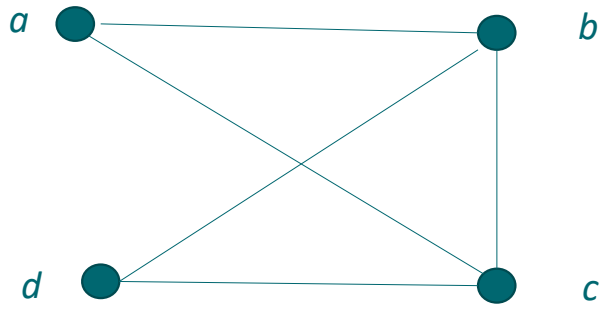
Graphs & Graph Models

Graphs

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- **Definition:** A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

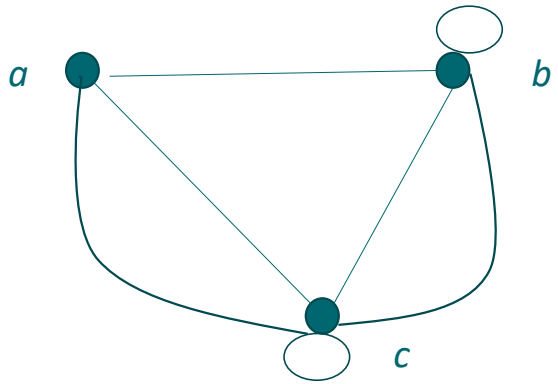
Example:
This is a graph with four vertices and five edges.



Some Terminology

- In a ***simple graph*** each edge connects two different vertices, and no two edges connect the same pair of vertices.
- ***Multigraphs*** may have multiple edges connecting the same two vertices. When *m* different edges connect the vertices *u* and *v*, we say that {*u, v*} is an edge of *multiplicity m*.
- An edge that connects a vertex to itself is called a ***loop***.
- A ***pseudograph*** may include loops, as well as multiple edges connecting the same pair of vertices.

Example:
This pseudograph has both multiple edges and a loop.



Remark: There is no standard terminology for graph theory. So, it is crucial that you understand the terminology being used whenever you read material about graphs.

Directed Graphs

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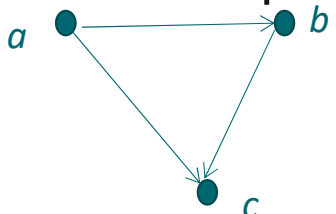
- **Definition:** A ***directed graph*** (or *digraph*) $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *directed edges* (or *arcs*). Each edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start at* u and *end at* v.

- **Remark**:
  - Graphs where the end points of an edge are not ordered are said to be *undirected graphs*.

Some Terminology (*continued*)

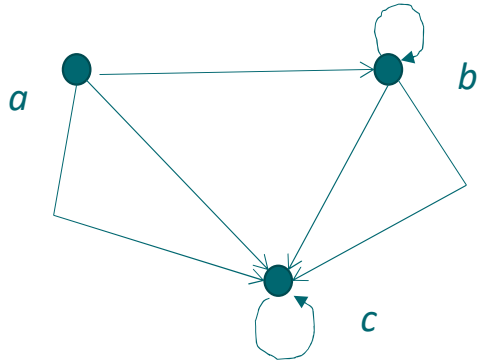
- A *simple directed graph* has no loops and no multiple edges.

Example:
This is a directed graph with three vertices and three edges.



- A *directed multigraph* may have multiple directed edges. When there are *m* directed edges from the vertex *u* to the vertex *v*, we say that (*u*, *v*) is an edge of *multiplicity m*.

Example:
In this directed multigraph the multiplicity of (*a*, *b*) is 1 and the multiplicity of (*b*, *c*) is 2.



Graph Terminology: Summary

- To understand the structure of a graph and to build a graph model, we ask these questions:
  - Are the edges of the graph undirected or directed (or both)?
  - If the edges are undirected, are multiple edges present that connect the same pair of vertices? If the edges are directed, are multiple-directed edges present?
  - Are loops present?

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Some Applicatons of Graph Theory

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Graph theory can be used in models of:

- Social networks
- Communications networks
- Information networks
- Software design
- Transportation networks
- Biological networks

It is a challenge to find a subject to which graph theory has not yet been applied. Can you find an area without applications of graph theory?





Graph Terminology
&
Special Types of Graphs

Basic Graph Terminology

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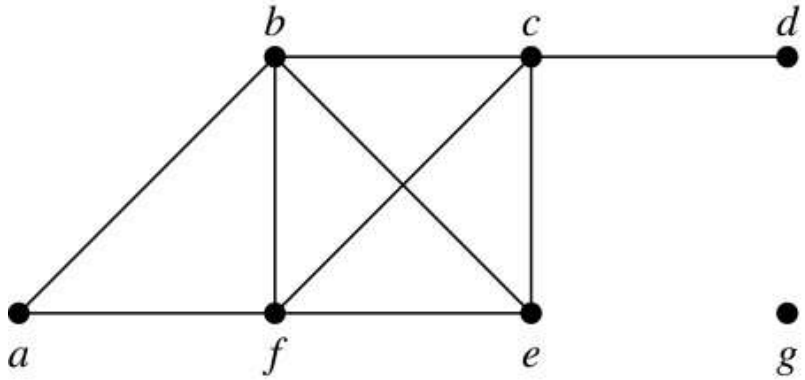
- **Definition 1**: Two vertices *u*, *v* in an undirected graph *G* are called *adjacent* (or *neighbors*) in *G* if there is an edge *e* between *u* and *v*. Such an edge *e* is called *incident with* the vertices *u* and *v* and *e* is said to *connect* *u* and *v*.
- **Definition 2**: The set of all neighbors of a vertex *v* of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of *v*. If *A* is a subset of *V*, we denote by $N(A)$ the set of all vertices in *G* that are adjacent to at least one vertex in *A*. So,

$$N(A) = \bigcup_{v \in A} N(v).$$

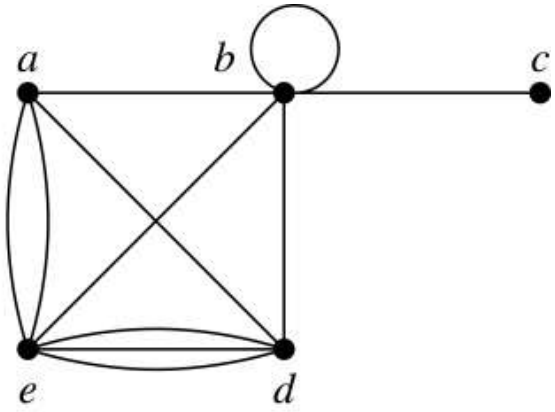
- **Definition 3**: The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex *v* is denoted by $\deg(v)$.

Degrees and Neighborhoods of Vertices

Example: What are the degrees and neighborhoods of the vertices in the graphs *G* and *H*?



G



H

Degrees and Neighborhoods of Vertices

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Solution:

G: $\deg(a) = 2, \deg(b) = \deg(c) = \deg(f) = 4, \deg(d)=1, \deg(e) = 3, \deg(g) = 0.$

$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, N(c) = \{b, d, e, f\}, N(d) = \{c\},$

$N(e) = \{b, c, f\}, N(f) = \{a, b, c, e\}, N(g) = \varnothing$.

H: $\deg(a) = 4, \deg(b) = \deg(e) = 6, \deg(c) = 1, \deg(d) = 5.$

$N(a) = \{b, d, e\}, N(b) = \{a, b, c, d, e\}, N(c) = \{b\},$

$N(d) = \{a, b, e\}, N(e) = \{a, b, d\}.$

Degrees of Vertices: Handshaking Theorem

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Theorem 1 (*Handshaking Theorem*): If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Handshaking Theorem

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Example: How many edges are there in a graph with 10 vertices of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, the handshaking theorem tells us that $2m = 60$. So, the number of edges $m = 30$.

Example: If a graph has 5 vertices, can each vertex have degree 3?

Solution: This is not possible by the handshaking theorem, because the sum of the degrees of the vertices $3 \cdot 5 = 15$ is odd.

Directed Graphs

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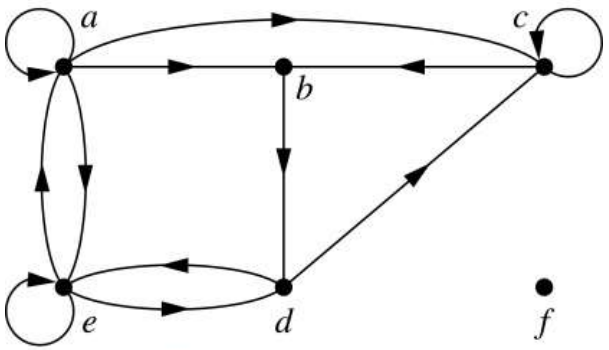
Definition: A *directed graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*), and E, a set of *directed edges* or *arcs*. Each edge is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.

Definition: Let (u, v) be an edge in G. Then u is the *initial vertex* of this edge and is *adjacent to* v and v is the *terminal* (or *end*) *vertex* of this edge and is *adjacent from* u. The initial and terminal vertices of a loop are the same.

Directed Graphs

Definition: The *in-degree of a vertex v*, denoted $deg^-(v)$, is the number of edges which terminate at *v*. The *out-degree of v*, denoted $deg^+(v)$, is the number of edges with *v* as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

Example: In the graph *G* we have



G

$\deg^-(a) = 2, \deg^-(b) = 2, \deg^-(c) = 3, \deg^-(d) = 2,$
$\deg^-(e) = 3, \deg^-(f) = 0.$

$\deg^+(a) = 4, \deg^+(b) = 1, \deg^+(c) = 2, \deg^+(d) = 2,$
$\deg^+(e) = 3, \deg^+(f) = 0.$

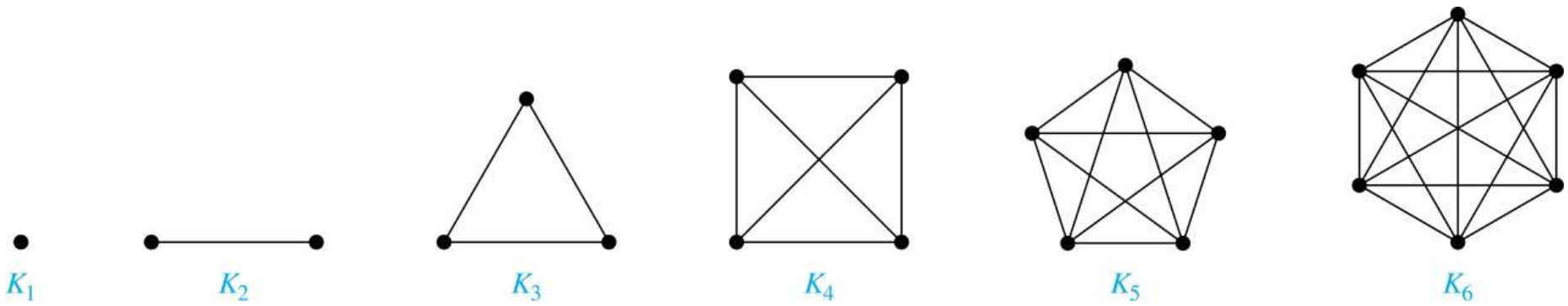
Directed Graphs (*continued*)

Theorem 3: Let $G = (V, E)$ be a graph with directed edges. Then:

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v).$$

Complete Graphs

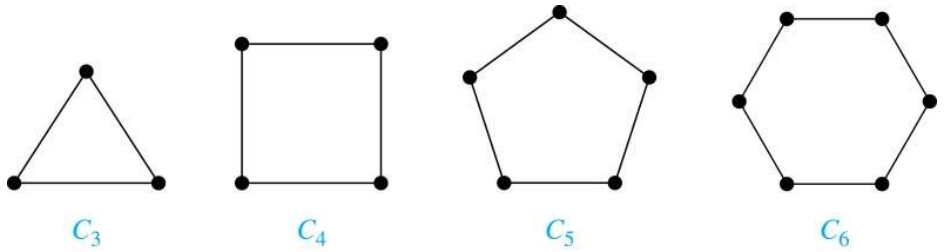
A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



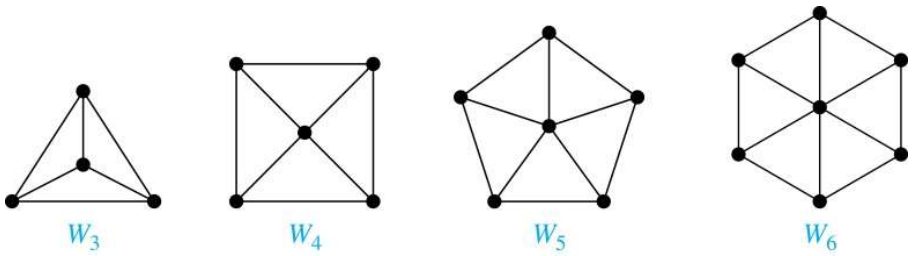
Special Types of Simple Graphs: Cycles and Wheels

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- A ***cycle*** C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



- A ***wheel*** W_n is obtained by adding an additional vertex to a cycle C_n for $n \geq 3$ and connecting this new vertex to each of the n vertices in C_n by new edges.

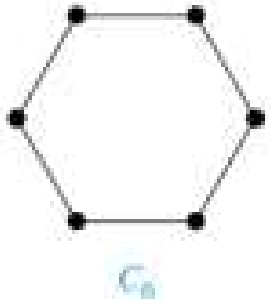


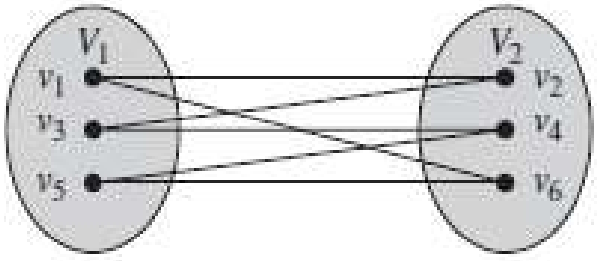
Bipartite Graphs

Definition: A simple graph G is *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2. When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Example:

C_6 is bipartite because its vertex set can be partitioned into the two sets $V_1 = \{v1, v3, v5\}$ and $V_2 = \{v2, v4, v6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

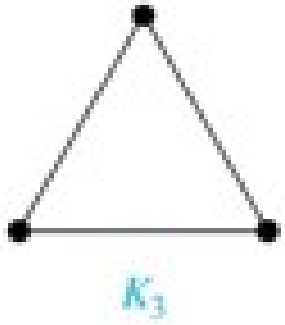




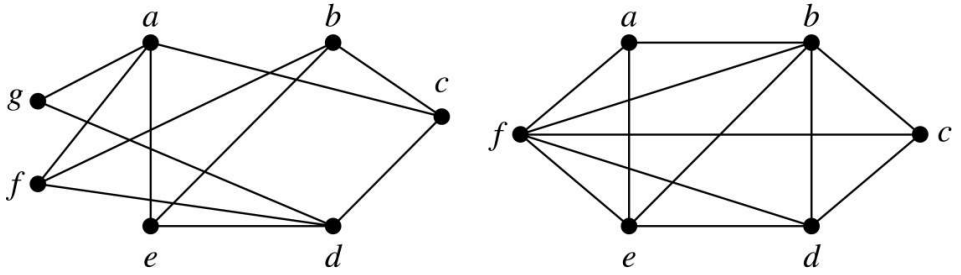
Bipartite Graphs (*continued*)

Example:

K_3 is not bipartite. To verify this, note that if we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.



Bipartite Graphs (*continued*)

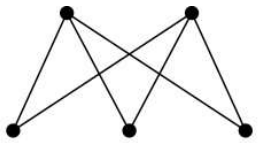


- *G* is bipartite because its vertex set is the union of two disjoint sets, {*a, b, d*} and {*c, e, f, g*}, and each edge connects a vertex in one of these subsets to a vertex in the other subset.
- (Note that for *G* to be bipartite it is not necessary that every vertex in {*a, b, d*} be adjacent to every vertex in {*c, e, f, g*}. For instance, *b* and *g* are not adjacent.)
- *H* is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. (considering the vertices *a*, *b*, and *f* .)

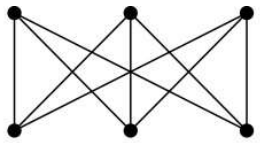
Complete Bipartite Graphs

Definition: A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

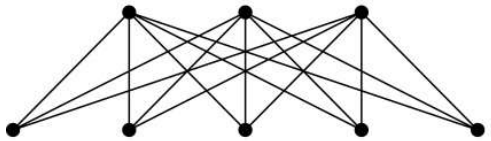
Example: We display four complete bipartite graphs here.



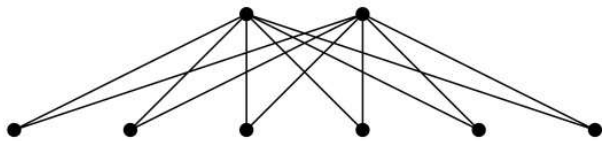
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

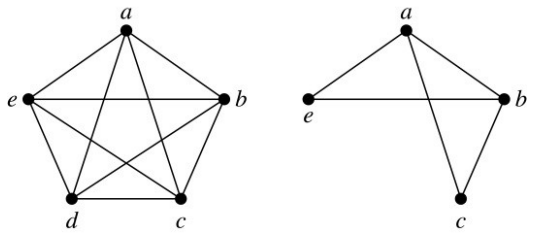


$K_{2,6}$

New Graphs from Old

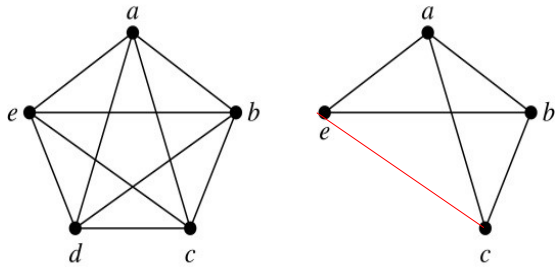
Definition: A ***subgraph*** *of a graph* $G = (V, E)$ is a graph (W,F), where $W \subset V$ and $F \subset E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

Example: Here we show K_5 and one of its subgraphs.



Definition: Let $G = (V, E)$ be a simple graph. The *subgraph* ***induced*** by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E if and only if both endpoints are in W.

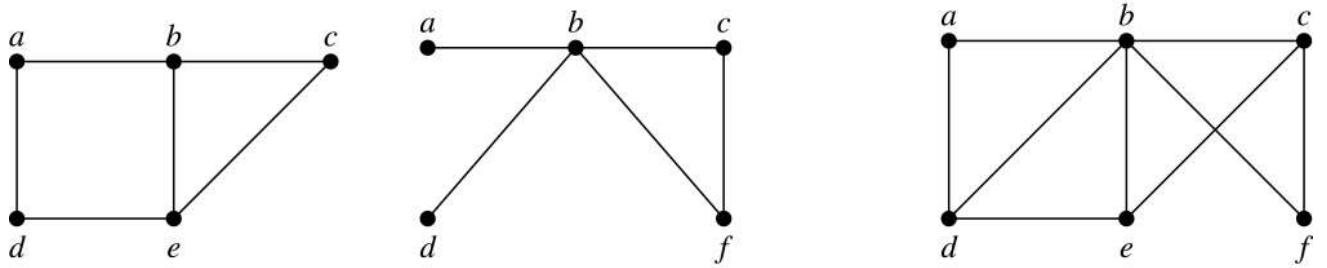
Example: Here we show K_5 and the subgraph induced by $W = \{a, b, c, e\}$.



New Graphs from Old (*continued*)

Definition: The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example:







Representing Graphs
&
Graph Isomorphism

Representing Graphs: Adjacency Lists

Definition: An *adjacency list* can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

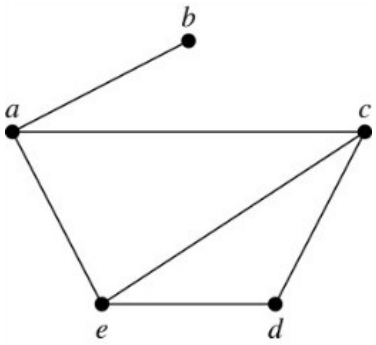


TABLE 1 An Adjacency List for a Simple Graph.

Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Example 1

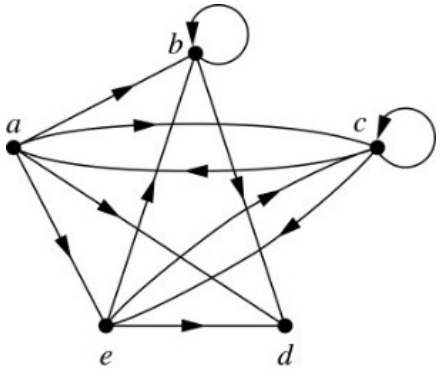


TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	Terminal Vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Example 2

Representing Graphs: Adjacency Matrices

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Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, with respect to the listing of vertices, is the $n \times n$ zero-one matrix with 1 as its $(i, j)^{\text{th}}$ entry when v_i and v_j are adjacent, and 0 as its $(i, j)^{\text{th}}$ entry when they are not adjacent.

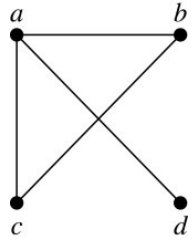
- In other words, if the graphs adjacency matrix is $\mathbf{A}_G = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency Matrices (continued)

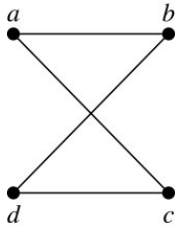
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Example:



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.

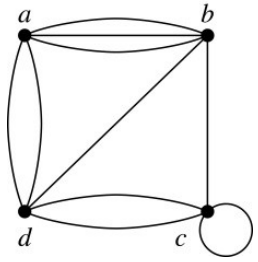
When a graph is sparse, that is, it has few edges relatively to the total number of possible edges, it is much more efficient to represent the graph using an adjacency list than an adjacency matrix. But for a dense graph, which includes a high percentage of possible edges, an adjacency matrix is preferable.

Note: The adjacency matrix of a simple graph is symmetric, i.e., $a_{ij} = a_{ji}$
Also, since there are no loops, each diagonal entry a_{ij} for $i = 1, 2, 3, \ldots, n$, is 0.

Adjacency Matrices (continued)

- Adjacency matrices can also be used to represent graphs with loops and multiple edges.
- A loop at the vertex v_i is represented by a 1 at the $(i, j)^{th}$ position of the matrix.
- When multiple edges connect the same pair of vertices v_i and v_j, (or if multiple loops are present at the same vertex), the $(i, j)^{th}$ entry equals the number of edges connecting the pair of vertices.

Example: We give the adjacency matrix of the pseudograph shown here using the ordering of vertices a, b, c, d.



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Adjacency Matrices (continued)

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- Adjacency matrices can also be used to represent directed graphs. The matrix for a directed graph $G = (V, E)$ has a 1 in its $(i, j)^{th}$ position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots v_n$ is a list of the vertices.
  - In other words, if the graphs adjacency matrix is $\mathbf{A}_G = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

  - The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from v_i to v_j, when there is an edge from v_j to v_i.
  - To represent directed multigraphs, the value of a_{ij} is the number of edges connecting v_i to v_j.

Isomorphism of Graphs

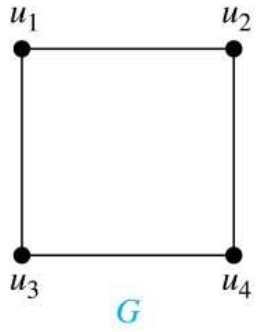
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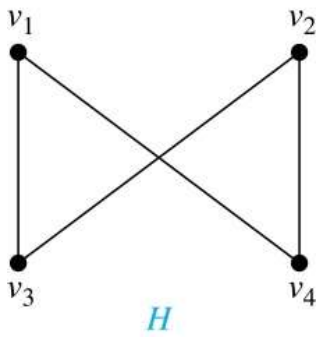
- Isomorphic simple graphs must have the **same** number of **vertices** and **edges**. In addition, the **degrees** of the vertices in isomorphic simple graphs must be the same.
- **Definition :** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 , for all a and b in V_1 . Such a function f is called an *isomorphism.*
- Two simple graphs that are not isomorphic are called *nonisomorphic*.

Isomorphism of Graphs (continued)

- **Example 1**: Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.

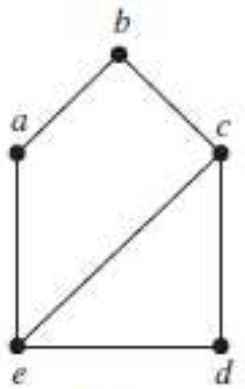
  - **Solution**:
  - The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$
    - is a one-to-one correspondence between V and W.
  - Adjacency:
    - G:{$u1$, $u2$}, {$u1$, $u3$}, {$u2$, $u4$} and {$u3$, $u4$}
    - *H:* each of the pairs consists of two adjacent vertices.
    - $f(u1)=v1$ and $f(u2)=v4$
    - $f(u1)=v1$ and $f(u3)=v3$
    - $f(u2)=v4$ and $f(u4)=v2$
    - $f(u3)=v3$ and $f(u4)=v2$
  - It follows that G and H are isomorphic.



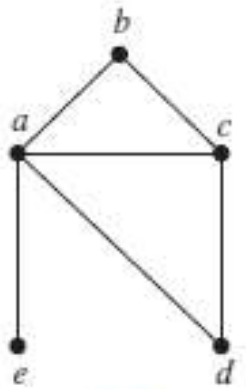


Isomorphism of Graphs (continued)

- **Example 2:** Determine whether these two graphs are isomorphic.
- **Solution:**
- Both G and H have five vertices and six edges.
- However, H has a vertex of degree one, namely, e,
- whereas G has no vertices of degree one.
- It follows that G and H are not isomorphic.



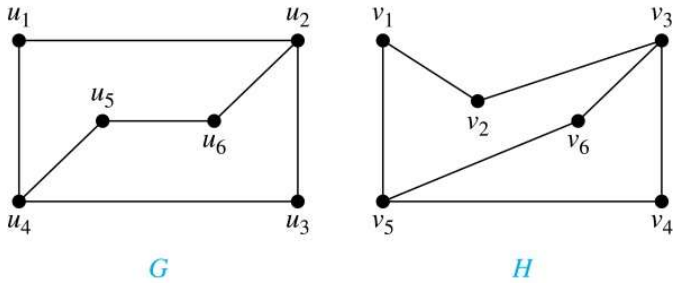
G



H

Isomorphism of Graphs (continued)

- **Example 3:** Determine whether these two graphs are isomorphic.
- **Solution:**
- Both *G* and *H* have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three.
- deg(u_1)= 2 and because u_1 is not adjacent to any other vertex of degree two, the image of u_1 must be either v_4 or v_6, the only vertices of degree two in *H* not adjacent to a vertex of degree two.
- $f(u_1)= \mathbf{v_6}$. [If we found that this choice did not lead to isomorphism, we would then try $f(u_1) = v_4$.]
- u_2 is adjacent to u_1, the possible images of u_2 are v_3 and v_5.
- $f(u_2)= \mathbf{v_3}$.
- $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5)= v_1$, and $f(u_6)= v_2$.



G *H*

What is Next?

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- **Read Sections 10.1, 10.2 and 10.3** from Rosen's Book (8th Ed).

- **Solve** the **Tutorial Exercises** sheet on BlackBoard (practical section) before the practical session.