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Discrete Structures 2

Graphs

Part 2

Computer Science and Artificial Intelligence Department
Umm Al-Qura University



Acknowledgment: Slides are mainly adopted from Rosen's book supplements.

AGENDA

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- **Graph Connectivity:**
  - Paths
  - Connectedness in Undirected Graphs
  - Connected Components
  - Connectedness in Directed Graphs
- **Euler and Hamiltonian Graphs:**
  - Euler Paths and Circuits
  - Hamilton Paths and Circuits





Graph Connectivity

Paths

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Informal Definition: A *path* is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these.

Applications: Numerous problems can be modeled with paths formed by traveling along edges of graphs such as:

- determining whether a message can be sent between two computers.
- efficiently planning routes for mail delivery.

Paths

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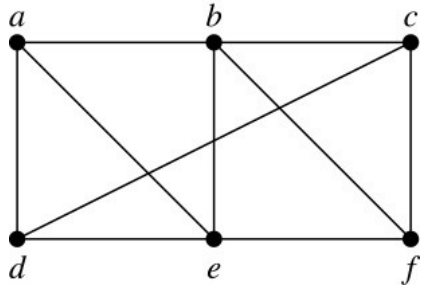
Definition: Let n be a nonnegative integer and G an undirected graph. A *path* of *length* n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the endpoints x_{i-1} and x_i.

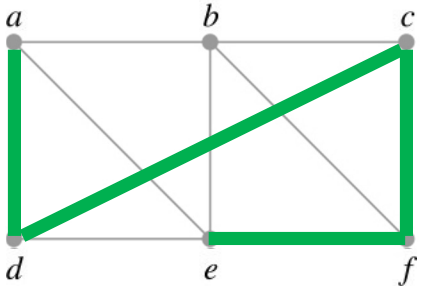
- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (since listing the vertices uniquely determines the path).
- The path is a *circuit* if it begins and ends at the same vertex ($u = v$) and has length greater than zero.
- The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ and *traverse* the edges $e_1, \ldots, e_n$.
- A path or circuit is *simple* if it does not contain the same edge more than once.

Paths

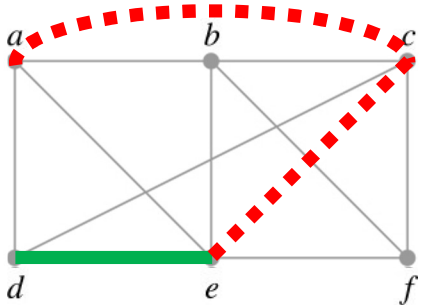
Example: In the simple graph here:

1. *a*, *d*, *c*, *f*, *e* is a simple path of length 4.
2. *d*, *e*, *c*, *a* is not a path because *e* is not connected to *c*.
3. *b*, *c*, *f*, *e*, *b* is a circuit of length 4.
4. *a*, *b*, *e*, *d*, *a*, *b* is a path of length 5, but it is not a simple path.

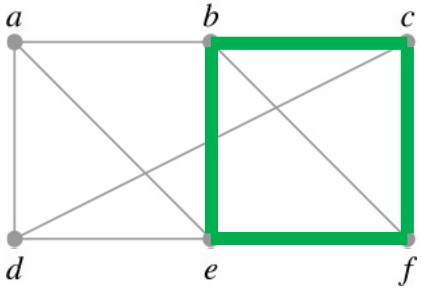




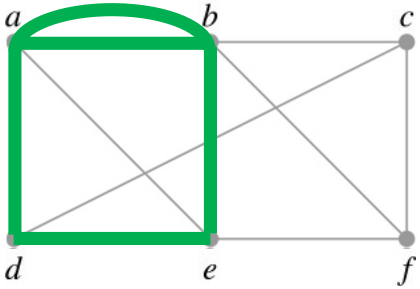
Simple path



Not a path



Circuit



Not simple path

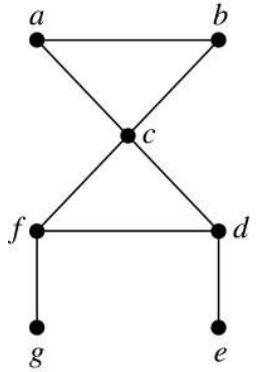
Connectedness in Undirected Graphs

Definition: An undirected graph is called *connected* if there is a path between every pair of vertices.

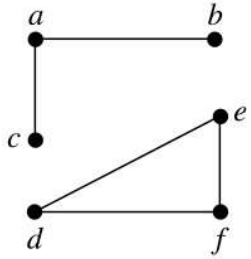
- An undirected graph that is not *connected* is called *disconnected*.
- We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Example:

- G_1 is connected because there is a path between any pair of its vertices, as can be easily seen.
- However, G_2 is not connected because there is no path between vertices *a* and *f*, for example.



G_1



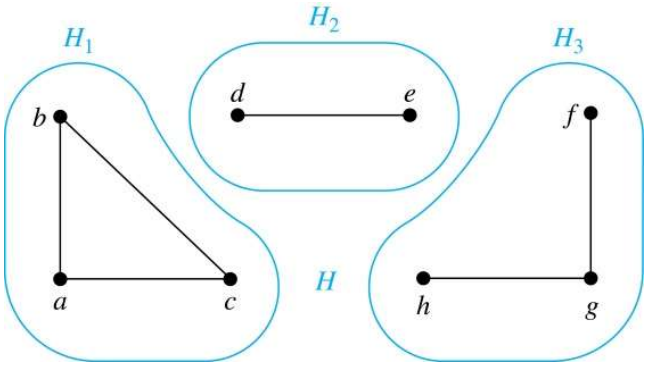
G_2

Connected Components

Definition: A *connected component* of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Example:

- The graph H is the union of three disjoint subgraphs H_1, H_2, and H_3, none of which are proper subgraphs of a larger connected subgraph of G.
- These three subgraphs are the connected components of H.



Connectedness in Directed Graphs

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Definition: A directed graph is *strongly connected* if there is a path from *a* to *b* and a path from *b* to *a* whenever *a* and *b* are vertices in the graph.

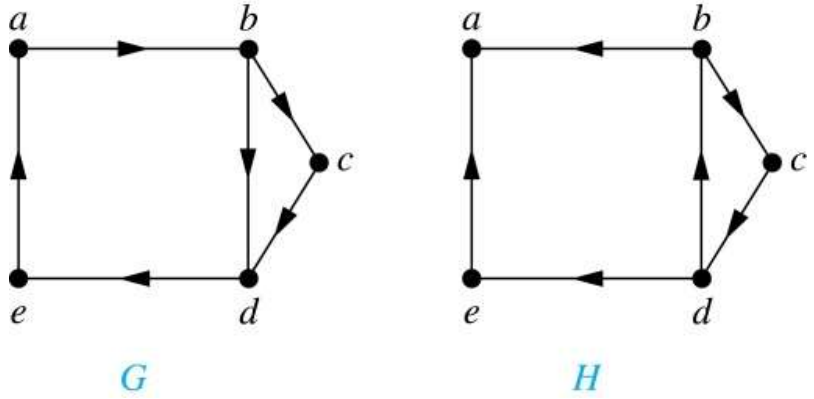
Definition: A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph, which is the undirected graph obtained by ignoring the directions of the edges of the directed graph.

Connectedness in Directed Graphs

Example:

G is strongly connected because there is a path between any two vertices in the directed graph.

Hence, *G* is also weakly connected. The graph *H* is not strongly connected, since there is no directed path from *a* to *b*, but it is weakly connected.







Euler
&
Hamiltonian Graphs

What is the difference between these questions?

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- Can we travel along the edges of a graph starting at a vertex and returning to it by **traversing each edge of the graph exactly once**?



Euler Circuit

- Can we travel along the edges of a graph starting at a vertex and returning to it while **visiting each vertex of the graph exactly once**?



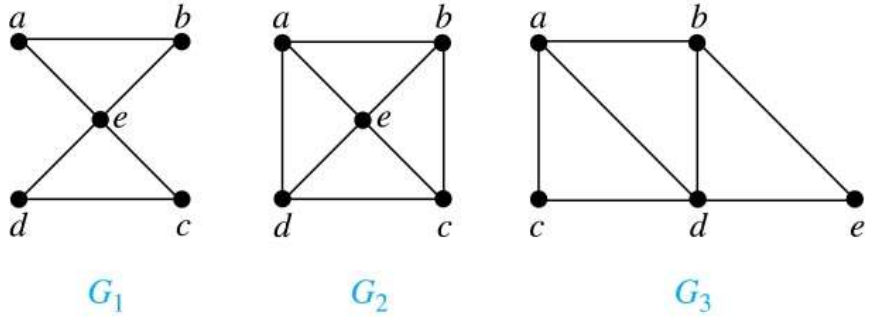
Hamilton Circuit

Euler Paths and Circuits (Undirected Graphs)

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Definition: An *Euler circuit* in an graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

Example:

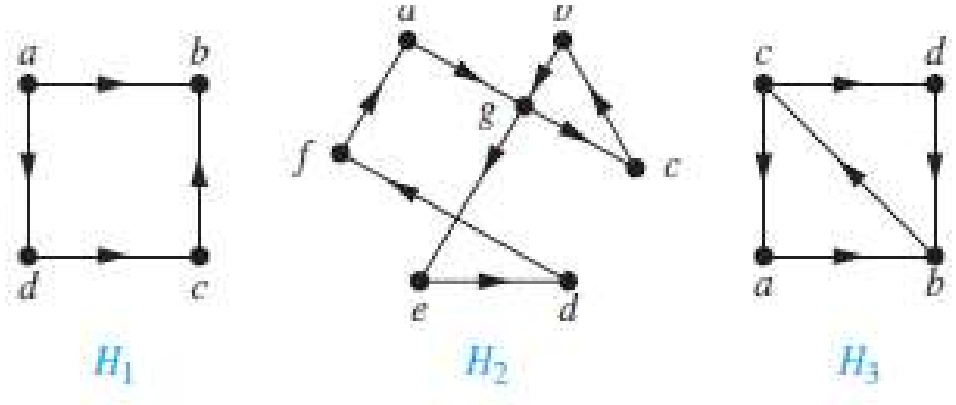


- The graph G_1 has an Euler circuit (e.g., a, e, c, d, e, b, a). Neither G_2 nor G_3 has an Euler circuit.
- G_3 has an Euler path (e.g., a, c, d, e, b, d, a, b).
- G_2 does not have an Euler path.

Euler Paths and Circuits (Directed Graphs)

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Example:



H_2 has an Euler circuit, for example, *a, g, c, b, g, e, d, f, a*.

Neither H_1 nor H_3 has an Euler circuit.

H_3 has an Euler path, namely, *c, a, b, c, d, b*, but H_1 does not.

Euler Paths and Circuits

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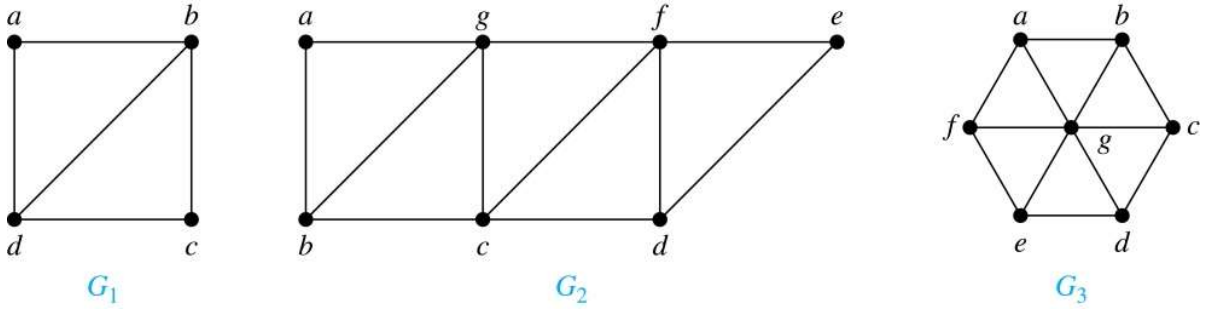
- **Theorem:** A connected multigraph with at least two vertices has an *Euler circuit* if and only if **each** of its vertices has **even** degree.

- **Theorem :** A connected multigraph has an *Euler path* but not an Euler circuit if and only if it has **exactly two** vertices of **odd** degree.
  - One is the start, the other is the end.

- Euler circuit in a **directed** graph: in-degrees must match out- degrees in all nodes.

Euler Paths and Circuits

Example:



- G_1 contains exactly two vertices of odd degree (*b* and *d*). Hence it has an Euler path, e.g., *d*, *a*, *b*, *c*, *d*, *b*.
- G_2 has exactly two vertices of odd degree (*b* and *d*). Hence it has an Euler path, e.g., *b*, *a*, *g*, *f*, *e*, *d*, *c*, *g*, *b*, *c*, *f*, *d*.
- G_3 has six vertices of odd degree. Hence, it does not have an Euler path.

Applications of Euler Paths and Circuits (FYI)

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- Euler paths and circuits can be used to solve many practical problems such as finding a path or circuit that traverses each
  - street in a neighborhood,
  - road in a transportation network,
  - connection in a utility grid,
  - link in a communications network.
- Other applications are found in the
  - layout of circuits,
  - network multicasting,
  - molecular biology, where Euler paths are used in the sequencing of DNA.

What is the difference between these questions?

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- Can we travel along the edges of a graph starting at a vertex and returning to it by **traversing each edge of the graph exactly once**?



Euler Circuit

- Can we travel along the edges of a graph starting at a vertex and returning to it while **visiting each vertex of the graph exactly once**?



Hamilton Circuit

Hamilton Paths and Circuits

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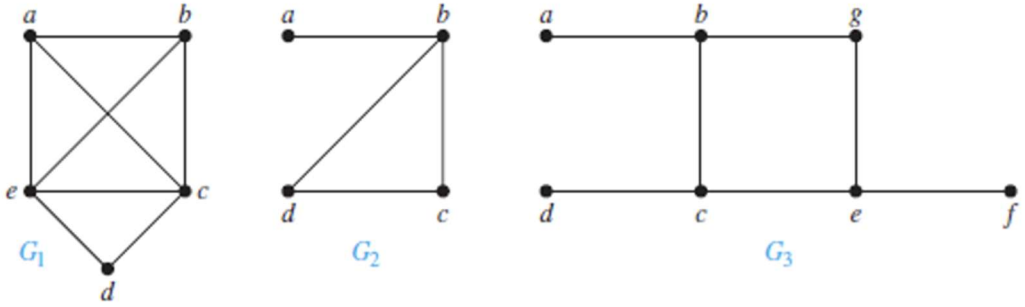
- Euler paths and circuits contained every edge only once. Now we look at paths and circuits that contain every vertex exactly once.

Definition: A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit.*

- That is, a simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is called a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

Hamilton Paths and Circuits

Example: Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



Solution:

- G_1 has a Hamilton circuit: *a*, *b*, *c*, *d*, *e*, *a*.
- G_2 does not have a Hamilton circuit (Why?), but does have a Hamilton path: *a*, *b*, *c*, *d*.
- G_3 does not have a Hamilton circuit, or a Hamilton path. Why?

Applications of Hamilton Paths and Circuits (FYI)

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- Applications that ask for a path or a circuit that visits each intersection of a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once, can be solved by finding a Hamilton path in the appropriate graph.

- The famous *traveling salesperson problem* (*TSP*) asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit such that the total sum of the weights of its edges is as small as possible.

What is Next?

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- **Read Sections 10.4 and 10.5** from Rosen's Book (8th Ed).

- **Solve** the **Tutorial Exercises** sheet on BlackBoard (practical section) before the practical session.