LINEAR ALGEBRA

Revision - Solved Edition

Prepared for Faisal

Original questions preserved, with an answer panel beside each page.

Includes:
117 multiple-choice answers with brief explanations
17 written analytical problems with worked solutions
17 analytical multiple-choice answers

How to use this copy
The side panel answers questions that START on that source page. If a question continues onto the next page, its answer stays beside its starting page.

Worked analytical solutions follow the original pages. Solutions use English to match the source document.

Source notes
The analytical heading says 18 questions, but only 17 are present. Printed page numbering skips page 22. All supplied pages are included.

Chapter 1 Question Bank

Multiple-Choice and Analytical Questions

Part I: Multiple-Choice Question Bank (63 Questions)

1. For vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, the dot product $v \cdot w$ is

A. $v_1 + w_1 + \cdots + v_n + w_n$

B. $v_1 w_1 + \cdots + v_n w_n$

C. $(v_1 w_1, \ldots, v_n w_n)$

D. $\|v\| + \|w\|$

2. If $v = (4, 2)$ and $w = (-1, 2)$, then $v \cdot w$ equals

A. -8

B. 0

C. 4

D. 8

3. Two nonzero vectors are perpendicular when

A. $v \cdot w = 1$

B. $v \cdot w = -1$

C. $v \cdot w = 0$

D. $\|v\| = \|w\|$

4. The squared length of a vector v is

A. $v \cdot v$

B. $v \cdot 0$

C. $\sqrt{v \cdot v}$

D. $2(v \cdot v)$

5. The length of $v = (1, 2, 3)$ is

A. 6

B. 14

C. $\sqrt{14}$

D. $\sqrt{6}$

6. A unit vector is a vector whose

A. $componentsareall1$

B. $lengthis1$

C. $dotproductwitheveryvectoris0$

ANSWERS & NOTES

Q1 - B
Add products of corresponding components.

Q2 - B
4(-1) + 2(2) = 0.

Q3 - C
Perpendicular nonzero vectors have dot product 0.

Q4 - A
The squared norm is v dot v.

Q5 - C
sqrt(1 + 4 + 9) = sqrt(14).

Q6 - B
A unit vector has length 1.

D. $lengthis0$

7. For nonzero v, a unit vector in the same direction as v is

A. $\|v\|v$
B. $v/\|v\|$
C. $v/(v \cdot v)^2$
D. $v + \|v\|$

8. If u and U are unit vectors making angle θ, then

A. $u \cdot U = \sin\theta$
B. $u \cdot U = \tan\theta$
C. $u \cdot U = \cos\theta$
D. $u \cdot U = \theta$

9. For nonzero vectors v, w, the cosine formula is

A. $\cos\theta = \dfrac{v \cdot w}{\|v\|\|w\|}$
B. $\cos\theta = (v \cdot w)\|v\|\|w\|$
C. $\cos\theta = \dfrac{\|v\|\|w\|}{v \cdot w}$
D. $\cos\theta = v \cdot w$

10. A linear equation in $x_1, \ldots, x_n$ has the form

A. $a_1x_1 + \cdots + a_nx_n = b$
B. $x_1x_2 = b$
C. $x_1^2 + x_2 = b$
D. $\sqrt{x_1} + x_2 = b$

11. Which equation is nonlinear?

A. $2x_1 - 3x_2 = 5$
B. $x_1 + x_2 - x_3 = 0$
C. $x_1x_2 = 4$
D. $7x_2 = 1$

12. A solution of a linear system is

A. a coefficient matrix
B. a list of values satisfying every equation
C. one equation from the system
D. a row operation

13. A linear system can have

ANSWERS & NOTES

Q7 - B
Divide v by its nonzero length.

Q8 - C
For unit vectors, the dot product equals cos(theta).

Q9 - A
cos(theta) = (v dot w)/(|v||w|).

Q10 - A
Only first-degree terms; no products of variables.

Q11 - C
The product x1*x2 makes the equation nonlinear.

Q12 - B
A solution satisfies every equation simultaneously.

Q13 - C
Zero, one, or infinitely many solutions.

A. only one solution

B. only zero or one solution

C. no solution, one solution, or infinitely many solutions

D. exactly two solutions

14. A consistent linear system has

A. no solution

B. at least one solution

C. exactly one solution only

D. infinitely many solutions only

15. An inconsistent linear system has

A. no solution

B. one solution

C. a free variable

D. a pivot in every coefficient column

16. The coefficient matrix records

A. only the right-hand constants

B. the coefficients of the variables

C. the solutions

D. only pivot entries

17. The augmented matrix is formed by adjoining to the coefficient matrix

A. an identity matrix

B. a row of zeros

C. the right-hand-side constants

D. the solution vector

18. A matrix with 3 rows and 4 columns has size

A. 4×3

B. 3×4

C. 7×1

D. 12×12

19. Which is an elementary row operation?

A. square every entry in a row

B. interchange two rows

C. add two columns

ANSWERS & NOTES

Q14 - B
Consistent means at least one solution.

Q15 - A
Inconsistent means no solution.

Q16 - B
The matrix contains variable coefficients.

Q17 - C
Append the right-hand-side constants.

Q18 - B
Size = rows x columns = 3 x 4.

Q19 - B
Interchanging two rows is allowed.

D. delete a zero row

20. In a scaling row operation, a row may be multiplied by

A. zero only

B. any nonzero constant

C. another row

D. a variable only

21. If two augmented matrices are row equivalent, their systems have

A. different solution sets

B. the same solution set

C. the same entries

D. the same number of pivots but never the same solutions

22. A row of the form $[0\ 0\ 0 \mid 5]$ in an augmented matrix implies

A. a unique solution

B. infinitely many solutions

C. inconsistency

D. a homogeneous system

23. In echelon form, each leading entry of a lower nonzero row lies

A. to the left of the leading entry above

B. in the same column as the leading entry above

C. to the right of the leading entry above

D. in the last column

24. Reduced echelon form additionally requires each leading entry to be

A. -1

B. 0

C. 1

D. any nonzero number

25. In reduced echelon form, each leading 1 is

A. the only nonzero entry in its column

B. the only nonzero entry in its row

C. always in the first column

D. always in the last row

26. A pivot position is a location corresponding to

A. a leading entry in an echelon form

ANSWERS & NOTES

Q20 - B
Scale a row by a nonzero constant.

Q21 - B
Elementary row operations preserve solutions.

Q22 - C
The row means 0 = 5, a contradiction.

Q23 - C
Leading entries move right as rows move down.

Q24 - C
Every leading entry in RREF is 1.

Q25 - A
A leading 1 is alone in its column.

Q26 - A
A pivot corresponds to an echelon leading entry.

B. any zero entry

C. the augmented column only

D. a free variable only

27. A pivot column of a matrix is a column that

A. contains a pivot position

B. contains only zeros

C. must be the last column

D. has no leading entry

28. The reduced echelon form of a matrix is

A. not unique

B. unique

C. unique only for square matrices

D. defined only for augmented matrices

29. A linear system is consistent if and only if an echelon form of its augmented matrix has

A. no row of the form $[0 \cdots 0 \mid b]$ with $b \neq 0$

B. a pivot in every variable column

C. no free variables

D. all entries nonzero

30. A variable corresponding to a pivot column is called

A. free

B. basic

C. inconsistent

D. augmented

31. A variable corresponding to a nonpivot variable column is called

A. basic

B. free

C. leading

D. constant

32. Two vectors in $\mathbb{R}^n$ are equal when

A. their lengths are equal

B. their corresponding entries are equal

C. their sums are zero

D. they have the same first entry

ANSWERS & NOTES

Q27 - A
A pivot column contains a pivot position.

Q28 - B
Every matrix has exactly one RREF.

Q29 - A
Consistency requires no contradictory row.

Q30 - B
Pivot variables are basic variables.

Q31 - A
Nonpivot variable columns give free variables.

Q32 - B
All corresponding components must match.

33. Vector addition is performed by

A. multiplying corresponding entries
B. adding corresponding entries
C. adding vector lengths only
D. interchanging entries

34. A scalar multiple cu is obtained by

A. adding c to each entry
B. multiplying each entry of u by c
C. dividing c by each entry
D. reordering entries

35. A linear combination of $v_1, \ldots, v_p$ has the form

A. $c_1v_1 + \cdots + c_pv_p$
B. $v_1 \cdot v_2 \cdots v_p$
C. $v_1 + \cdots + v_p$ only
D. $c_1 + \cdots + c_p$

36. Span$\{v_1, \ldots, v_p\}$ is the set of

A. only the listed vectors
B. all linear combinations of the listed vectors
C. all unit vectors
D. all vectors perpendicular to them

37. For $A = [a_1\ a_2\ \cdots\ a_n]$ and $x \in \mathbb{R}^n$, Ax equals

A. $x_1a_1 + \cdots + x_na_n$
B. $a_1 + \cdots + a_n$
C. $x_1 + \cdots + x_n$
D. $A + x$

38. The product Ax is defined when

A. the number of rows of A equals entries in x
B. the number of columns of A equals entries in x
C. A is square
D. x is a unit vector

39. The matrix equation $Ax = b$ and the corresponding vector equation have

A. different solution sets
B. the same solution set

ANSWERS & NOTES

Q33 - B
Add corresponding components.

Q34 - B
Multiply every component by c.

Q35 - A
A linear combination is a weighted sum of vectors.

Q36 - B
Span contains all linear combinations.

Q37 - A
Ax is the sum of columns weighted by x.

Q38 - B
Columns of A must equal the number of entries in x.

Q39 - B
Both equations express the same constraints.

C. the same number of rows only

D. no relation

40. For an $m \times n$ matrix A, the columns of A span $\mathbb{R}^m$ if and only if

A. A has a pivot in every row

B. A has a pivot in every column

C. $m = n$

D. A has no zeros

41. Matrix–vector multiplication satisfies

A. $A(u + v) = Au + Av$

B. $A(u + v) = Au + v$

C. $A(cu) = Au + c$

D. $A0 = A$

42. Matrix–vector multiplication also satisfies

A. $A(cu) = c(Au)$

B. $A(cu) = c + Au$

C. $A(cu) = A + cu$

D. $A(cu) = 0$ always

43. The equation $Ax = 0$ is called

A. nonhomogeneous

B. homogeneous

C. inconsistent

D. nonlinear

44. Every homogeneous system $Ax = 0$ is

A. inconsistent

B. consistent

C. unique

D. nonlinear

45. The solution $x = 0$ of $Ax = 0$ is called

A. a free solution

B. the trivial solution

C. a pivot solution

D. an augmented solution

46. A nontrivial solution of $Ax = 0$ is

ANSWERS & NOTES

Q40 - A
A pivot in every row means the columns span R^m.

Q41 - A
Matrix-vector multiplication distributes over addition.

Q42 - A
A(cu) = c(Au).

Q43 - B
Zero right-hand side means homogeneous.

Q44 - B
x = 0 is always a solution.

Q45 - B
x = 0 is the trivial solution.

Q46 - A
Nontrivial means x is not the zero vector.

A. any solution other than $x = 0$

B. only a unit vector

C. always unique

D. impossible

47. The homogeneous equation $Ax = 0$ has a nontrivial solution if

A. there is a free variable

B. there are no variables

C. the system is inconsistent

D. the augmented column is a pivot column

48. A parametric vector form expresses a solution vector using

A. free variables as parameters

B. only pivot positions

C. no vectors

D. determinants

49. A set $\{v_1, \ldots, v_p\}$ is linearly independent when

A. $x_1v_1 + \cdots + x_pv_p = 0$ has only the trivial solution

B. it contains the zero vector

C. one vector is a combination of the others

D. $p > n$ in $\mathbb{R}^n$

50. A set is linearly dependent when there exist scalars, not all zero, such that

A. $c_1v_1 + \cdots + c_pv_p = 0$

B. $c_1 + \cdots + c_p = 1$

C. $v_1 + \cdots + v_p = 1$

D. all vectors have length 1

51. The columns of A are linearly independent if and only if

A. $Ax = 0$ has only the trivial solution

B. $Ax = b$ is inconsistent

C. A has a zero row

D. A has more columns than rows

52. Two nonzero vectors are linearly dependent if and only if

A. they are perpendicular

B. one is a scalar multiple of the other

C. they have equal lengths

ANSWERS & NOTES

Q47 - A
A free variable gives nontrivial homogeneous solutions.

Q48 - A
Free variables serve as parameters.

Q49 - A
Independence means only the trivial relation.

Q50 - A
Dependence requires a nonzero coefficient combination giving 0.

Q51 - A
Column independence means Ax = 0 has only x = 0.

Q52 - B
For two nonzero vectors, dependence means proportionality.

D. their dot product is positive

53. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the columns are

A. linearly dependent

B. linearly independent

C. zero vectors

D. not in $\mathbb{R}^2$

54. For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, the columns are

A. linearly independent

B. linearly dependent

C. perpendicular

D. unit vectors

55. The system $x_1 + x_2 = 2$, $2x_1 + 2x_2 = 4$ has

A. no solution

B. exactly one solution

C. infinitely many solutions

D. exactly two solutions

56. The system $x_1 + x_2 = 2$, $2x_1 + 2x_2 = 5$ is

A. consistent

B. inconsistent

C. homogeneous

D. uniquely solvable

57. The system $x_1 = 3$, $x_2 = -1$ has

A. a unique solution

B. infinitely many solutions

C. no solution

D. a free variable

58. If an echelon form of a coefficient matrix has a pivot in every column, then $Ax = 0$ has

A. a free variable

B. only the trivial solution

C. no solution

D. an augmented pivot

ANSWERS & NOTES

Q53 - B
The identity columns are independent.

Q54 - A
Column 2 is twice column 1.

Q55 - C
The second equation is twice the first.

Q56 - A
Doubling the first equation would give 4, not 5.

Q57 - A
Both variables have fixed values.

Q58 - B
Every variable is basic, so x = 0 is the only solution.

59. If an $m \times n$ matrix has a pivot in every row, then its columns

A. span $\mathbb{R}^m$

B. are necessarily linearly independent

C. must be zero

D. cannot span $\mathbb{R}^m$

60. If an $m \times n$ matrix has a pivot in every column, then its columns are

A. linearly independent

B. necessarily spanning $\mathbb{R}^m$

C. linearly dependent

D. all unit vectors

61. A 3×5 matrix can have at most

A. 2 pivots

B. 3 pivots

C. 5 pivots

D. 8 pivots

62. A 5×3 matrix with a pivot in each of its three columns has columns that are

A. linearly independent

B. linearly dependent

C. spanning $\mathbb{R}^5$

D. zero

63. A 5×3 matrix cannot have columns that span $\mathbb{R}^5$ because it can have at most

A. 3 pivot positions

B. 5 pivot positions

C. 8 pivot positions

D. 1 pivot position

ANSWERS & NOTES

Q59 - A
A pivot in every row gives spanning.

Q60 - A
A pivot in every column gives independence.

Q61 - B
At most min(3,5) = 3 pivots.

Q62 - A
Every column is a pivot column.

Q63 - A
At most 3 pivots cannot cover 5 rows.

Part II: Additional Computational MCQ Bank (54 Questions)

64. If $u = (1, 2)$ and $v = (3, -1)$, then $u + v$ is

A. $(4, 1)$
B. $(3, 2)$
C. $(2, -3)$
D. $(4, -2)$

65. If $u = (2, -3)$, then $-2u$ is

A. $(-4, 6)$
B. $(4, -6)$
C. $(0, -5)$
D. $(-1, 3/2)$

66. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, Ax is

A. $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
B. $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$
C. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
D. $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

67. For $v = (3, 4)$, $\|v\|$ equals

A. 5
B. 7
C. 25
D. $\sqrt{7}$

68. For $v = (1, 0)$ and $w = (0, 1)$, the angle between them is

A. $0°$
B. $45°$
C. $90°$
D. $180°$

69. Let $u = (2, -3)$ and $v = (4, 1)$. Then $u \cdot v$ is

A. 5

ANSWERS & NOTES

Q64 - A
(1+3, 2-1) = (4,1).

Q65 - A
-2(2,-3) = (-4,6).

Q66 - A
(2-2, 6-4) = (0,2).

Q67 - A
sqrt(9+16) = 5.

Q68 - C
Dot product 0 gives 90 degrees.

Q69 - A
2(4) + (-3)(1) = 5.

B. 8

C. 11

D. -11

70. If $v = (-2, 3, 6)$, then $\|v\|$ is

A. 7

B. 49

C. $\sqrt{13}$

D. 11

71. A unit vector in the direction of $v = (3, 4)$ is

A. $\left(\frac{3}{5}, \frac{4}{5}\right)$

B. $\left(\frac{4}{3}, \frac{3}{4}\right)$

C. $(5, 5)$

D. $\left(\frac{1}{5}, \frac{1}{5}\right)$

72. For $u = (1, 2)$ and $v = (2, -1)$, the angle between u and v is

A. $0°$

B. $45°$

C. $90°$

D. $180°$

73. For $u = (1, 0)$ and $v = (1, \sqrt{3})$, the angle between them is

A. $30°$

B. $45°$

C. $60°$

D. $120°$

74. For $u = (1, 2)$ and $v = (3, 4)$, the quantity $\|u\|\,\|v\|$ equals

A. 11

B. $5\sqrt{5}$

C. $\sqrt{11}$

D. 25

75. If $u = (1, 2)$ and $v = (2, 1)$, then $\|u + v\|$ is

A. 3

B. $3\sqrt{2}$

C. $\sqrt{5}$

D. 6

ANSWERS & NOTES

Q70 - A
sqrt(4+9+36) = 7.

Q71 - A
(3,4)/5 = (3/5,4/5).

Q72 - C
1(2)+2(-1) = 0; angle = 90 degrees.

Q73 - C
cos(theta) = 1/(1*2) = 1/2; theta = 60 degrees.

Q74 - B
sqrt(5)*5 = 5sqrt(5).

Q75 - B
u+v = (3,3); length = sqrt(18) = 3sqrt(2).

76. Solve $x + y = 5,\ x - y = 1$. The solution is

A. $(2, 3)$
B. $(3, 2)$
C. $(4, 1)$
D. $(1, 4)$

77. Solve $2x + y = 7,\ x - y = 2$. The value of x is

A. 2
B. 3
C. 4
D. 5

78. The system $x + 2y = 4,\ 2x + 4y = 8$ has

A. no solution
B. exactly one solution
C. infinitely many solutions
D. exactly two solutions

79. The system $x + 2y = 4,\ 2x + 4y = 9$ has

A. no solution
B. exactly one solution
C. infinitely many solutions
D. the solution $(0, 2)$

80. For what value of h is $x + 3y = 2,\ 2x + 6y = h$ consistent?

A. $h = 2$
B. $h = 3$
C. $h = 4$
D. $h = 6$

81. The augmented matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$ represents a system whose solution is

A. $(-1, 3)$
B. $(3, -1)$
C. $(5, 3)$
D. $(1, 2)$

82. After $R_2 \leftarrow R_2 - 2R_1$ on $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$, the second row becomes

A. [0 1 2]

ANSWERS & NOTES

Q76 - B
Add equations: 2x=6. Thus x=3, y=2.

Q77 - B
Add equations: 3x=9. Thus x=3.

Q78 - C
Second equation = twice the first; infinitely many solutions.

Q79 - C
Subtract twice the first: 0=1, so no solution.

Q80 - C
Consistency requires h=2(2)=4.

Q81 - A
y=3; x=5-2(3)=-1.

Q82 - A
[2,5,8]-2[1,2,3] = [0,1,2].

B. $[0\ 1\ 5]$

C. $[1\ 3\ 5]$

D. $[0\ 3\ 2]$

83. The reduced echelon form of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is

A. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

84. For $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the pivot columns are

A. $1, 2$

B. $1, 3$

C. $2, 4$

D. $3, 4$

85. For the augmented matrix $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 1 \end{bmatrix}$, if $x_3 = 2$, then (x_1, x_2) equals

A. $(1, 7)$

B. $(9, -5)$

C. $(1, -5)$

D. $(5, 1)$

86. The augmented matrix $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ represents a system that is

A. consistent with one solution

B. consistent with infinitely many solutions

C. inconsistent

D. homogeneous

87. A consistent system has 5 variables and 3 pivot columns in its coefficient matrix. The number of free variables is

A. 1

ANSWERS & NOTES

Q83 - A
R2 <- R2-2R1 gives a zero second row.

Q84 - B
Leading entries lie in columns 1 and 3.

Q85 - A
x3=2; x1=5-2(2)=1; x2=1+3(2)=7.

Q86 - C
Last row means 0=1.

Q87 - B
Free variables = 5-3 = 2.

B. 2

C. 3

D. 5

88. If the reduced row echelon form is $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -1 \end{bmatrix}$, then with $x_3 = t$ the solution is

A. $(3 + 2t, -1 - 4t, t)$

B. $(3 - 2t, -1 + 4t, t)$

C. $(3 + 2t, -1 + 4t, t)$

D. $(3 - 2t, -1 - 4t, t)$

89. If $u = (1, 2)$ and $v = (3, -1)$, then $2u - v$ is

A. $(-1, 5)$

B. $(5, 1)$

C. $(-1, 3)$

D. $(1, 5)$

90. If $u = (1, 0, 2)$, $v = (0, 1, -1)$, and $w = 2u + 3v$, then w is

A. $(2, 3, 1)$

B. $(2, 3, 7)$

C. $(3, 2, 1)$

D. $(2, -3, 1)$

91. Let $u = (1, 2)$ and $v = (2, -1)$. If $c_1u + c_2v = (5, 0)$, then (c_1, c_2) is

A. $(1, 2)$

B. $(2, 1)$

C. $(1, 1)$

D. $(2, 2)$

92. Which pair (a, b) makes $a(1, 0, 1) + b(0, 1, 1) = (2, 3, 5)$?

A. $(2, 3)$

B. $(3, 2)$

C. $(2, 5)$

D. $(5, 3)$

93. Which vector is *not* in $\text{Span}\{(1, 0, 1), (0, 1, 1)\}$?

A. $(1, 1, 2)$

B. $(2, -1, 1)$

C. $(3, 4, 7)$

D. $(1, 2, 4)$

ANSWERS & NOTES

Q88 - A
x1=3+2t; x2=-1-4t; x3=t.

Q89 - A
2(1,2)-(3,-1) = (-1,5).

Q90 - A
2(1,0,2)+3(0,1,-1) = (2,3,1).

Q91 - A
c1+2c2=5, 2c1-c2=0; (c1,c2)=(1,2).

Q92 - A
First two components give a=2, b=3; third is 5.

Q93 - D
Every span vector has third component = first + second; 4 != 1+2.

94. If $u = (2, 4)$, then the value of t for which $tu = (6, 12)$ is

A. 2

B. 3

C. 4

D. 6

95. For $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, Ax equals

A. $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

C. $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$

D. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

96. For $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, Ax is

A. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

97. If $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, then the solution of $Ax = b$ is

A. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

98. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $A(u + v)$ is

ANSWERS & NOTES

Q94 - B
2t=6 gives t=3.

Q95 - A
(2*1+4*2, 2*3-4) = (10,2).

Q96 - A
(2+0-2, -2+3-1) = (0,0).

Q97 - B
2x+y=5 and x-y=1 give (x,y)=(2,1).

Q98 - A
u+v=(3,0); A(u+v)=(3,0).

A. $\begin{bmatrix}3\\0\end{bmatrix}$

B. $\begin{bmatrix}3\\3\end{bmatrix}$

C. $\begin{bmatrix}5\\0\end{bmatrix}$

D. $\begin{bmatrix}5\\3\end{bmatrix}$

99. For $A = \begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}$ and $u = \begin{bmatrix}3\\1\end{bmatrix}$, $A(2u)$ is

A. $\begin{bmatrix}5\\-5\end{bmatrix}$

B. $\begin{bmatrix}10\\-10\end{bmatrix}$

C. $\begin{bmatrix}6\\2\end{bmatrix}$

D. $\begin{bmatrix}8\\-8\end{bmatrix}$

100. For $A = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\2 & 0 & 1\end{bmatrix}$ and $x = \begin{bmatrix}1\\-1\\2\end{bmatrix}$, the third entry of Ax is

A. 0

B. 2

C. 4

D. 5

101. If A is 3×5 and $x \in \mathbb{R}^5$, then Ax has

A. 3 entries

B. 5 entries

C. 8 entries

D. 15 entries

102. Solve $x_1 - 2x_2 = 0$. If $x_2 = 3$, then x_1 is

A. -6

B. -3

C. 3

D. 6

103. For $x_1 + 2x_2 - x_3 = 0$, if $x_2 = s$ and $x_3 = t$, then x_1 is

ANSWERS & NOTES

Q99 - B
Au=(5,-5); A(2u)=(10,-10).

Q100 - C
Third row dot x = 2(1)+0(-1)+1(2)=4.

Q101 - A
A maps a 5-entry vector to a 3-entry vector.

Q102 - D
x1=2x2=2(3)=6.

Q103 - A
x1=-2s+t.

A. $-2s + t$

B. $2s - t$

C. $-2s - t$

D. $2s + t$

104. The solution set of $x_1 - 3x_2 = 0$ can be written $x = t\,v$. Which vector can be used for v?

A. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

105. If all solutions of $Ax = 0$ are $x = t \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, then for $t = -2$ the solution is

A. $\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$

B. $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix}$

D. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

106. If $p = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $x = p + tv$, then for $t = -1$, x is

A. $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$

ANSWERS & NOTES

Q104 - B
x1=3x2; x=t(3,1).

Q105 - A
-2(2,-1,3)=(-4,2,-6).

Q106 - A
p-v=(-1,2,0)-(2,1,-1)=(-3,1,1).

D. $\begin{bmatrix}1\\1\\1\end{bmatrix}$

107. The Reduced Row Echelon Form of a coefficient matrix for a homogeneous system with 4 variables has pivots in columns 1 and 3. The number of free variables is

A. 1

B. 2

C. 3

D. 4

108. The vectors $v_1 = (1, 2)$ and $v_2 = (3, 6)$ satisfy $c_1v_1 + c_2v_2 = 0$ for which nontrivial pair?

A. $(3, -1)$

B. $(1, 1)$

C. $(2, 1)$

D. $(1, -1)$

109. For $v_1 = (1, 0)$ and $v_2 = (0, 1)$, $c_1v_1 + c_2v_2 = 0$ has

A. $c_1 = 1, c_2 = 1$

B. $c_1 = 1, c_2 = -1$

C. $c_1 = 0, c_2 = 0$ only

D. infinitely many nontrivial solutions

110. Let $v_1 = (1, 2, 3)$, $v_2 = (2, 4, 6)$, $v_3 = (0, 1, 1)$. Which relation proves dependence?

A. $2v_1 - v_2 = 0$

B. $v_1 + v_2 = 0$

C. $v_1 - v_3 = 0$

D. $v_2 - v_3 = 0$

111. For $A = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 3\\0 & 0 & 4\end{bmatrix}$, the equation $Ax = 0$ has

A. one free variable

B. two free variables

C. only the trivial solution

D. infinitely many nontrivial solutions

112. For $A = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\end{bmatrix}$, a nontrivial solution of $Ax = 0$ is

A. $\begin{bmatrix}5\\-4\\1\end{bmatrix}$

ANSWERS & NOTES

Q107 - B
4 variables - 2 pivots = 2 free variables.

Q108 - A
3(1,2)-(3,6)=(0,0).

Q109 - C
c1(1,0)+c2(0,1)=0 forces both coefficients to be 0.

Q110 - A
v2=2v1, hence 2v1-v2=0.

Q111 - C
Three pivots, no free variables: only x=0.

Q112 - A
x3=t; x2=-4t; x1=5t.

B. $\begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}$

C. $\begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$

113. If $v_1 = (1, 1)$, $v_2 = (2, -1)$, and $v_3 = (4, 1)$, then $v_3 = c_1 v_1 + c_2 v_2$ with

A. $c_1 = 1, c_2 = 2$

B. $c_1 = 2, c_2 = 1$

C. $c_1 = 3, c_2 = 1$

D. $c_1 = 1, c_2 = 3$

114. For $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, a nontrivial solution of $Ax = 0$ is

A. $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

115. For $x_1 + x_2 + x_3 = 4,\ x_2 + 2x_3 = 3$, if $x_3 = 1$, the solution vector is

A. $(2, 1, 1)$

B. $(1, 2, 1)$

C. $(3, 0, 1)$

D. $(1, 1, 2)$

116. For $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, if $x_3 = 1$, the corresponding solution of $Ax = b$ is

A. $(1, 2, 1)$

B. $(3, 2, 1)$

ANSWERS & NOTES

Q113 - B
2(1,1)+(2,-1)=(4,1).

Q114 - A
x3=t; x1=-2t; x2=t.

Q115 - A
x3=1; x2=3-2=1; x1=4-1-1=2.

Q116 - A
x3=1; x2=5-3=2; x1=4-2(2)+1=1.

C. $(1, -2, 1)$

D. $(5, 2, 1)$

117. Row reducing $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ gives two pivot columns. Hence the number of free variables in $Ax = 0$ is

A. 0

B. 1

C. 2

D. 3

ANSWERS & NOTES

Q117 - B

3 variables - 2 pivots = 1 free variable.

Part IV: Analytical Question Bank (18 Questions)

1. Solve the system by elementary row operations and write the complete solution set in parametric vector form. Verify your result by substitution:

$$x_1 + 2x_2 - x_3 = 3, \qquad 2x_1 - x_2 + 3x_3 = 9, \qquad 3x_1 + x_2 + 2x_3 = 12.$$

2. Determine whether the system is consistent. If it is consistent, state whether the solution is unique or infinite:

$$x_1 - 2x_2 + x_3 = 1, \quad 2x_1 - 4x_2 + 2x_3 = 2, \quad 3x_1 - 6x_2 + 3x_3 = 5.$$

3. Find the reduced echelon form of

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & 9 \\ -1 & -2 & 2 & -4 \end{bmatrix}.$$

Identify all pivot positions and pivot columns.

4. For the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

identify the basic and free variables and write the complete solution set in parametric vector form.

5. Find all values of h for which the system

$$x_1 + 2x_2 = 3, \qquad 2x_1 + 4x_2 = h$$

is consistent. For those values, describe the solution set.

6. Construct one example each of a linear system with (a) no solution, (b) exactly one solution, and (c) infinitely many solutions. Explain how the echelon form reveals each case.

7. Let $v = (2, -1, 3)$ and $w = (1, 4, -2)$. Compute $v \cdot w$, $\|v\|$, $\|w\|$, and the angle between the vectors.

8. Normalize the vector $v = (2, -2, 1)$. Verify directly that the resulting vector has length 1.

9. Find a nonzero vector perpendicular to $v = (2, -1, 3)$, and verify your answer using the dot product.

10. Express $b = (7, 4, -3)$ as a linear combination of $a_1 = (1, -2, -5)$ and $a_2 = (2, 5, 6)$, or prove that this is impossible.

11. Determine whether $b = (1, 2, 4)$ belongs to $\text{Span}\{(1, 0, 1), (0, 1, 1)\}$. Justify by solving the corresponding vector equation.

ANSWERS & NOTES

WRITTEN PROBLEMS 1-11
See the worked solutions after the original pages.

Source heading says 18 questions; only 17 are supplied.

12. Write the vector equation $x_1a_1 + x_2a_2 + x_3a_3 = b$ as a matrix equation for $a_1 = (1, 2)$, $a_2 = (-1, 3)$, $a_3 = (4, 0)$, and $b = (5, 1)$, then solve it.

13. For

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \\ 2 & 1 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix},$$

compute Ax both as a linear combination of the columns and by the row–vector rule.

14. Row reduce

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

Do its columns span $\mathbb{R}^3$? Explain using pivot positions.

15. Solve $Ax = 0$ for

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 \end{bmatrix}$$

and write the solution set in parametric vector form.

16. Determine whether $v_1 = (1, 2, 3)$, $v_2 = (4, 5, 6)$, and $v_3 = (2, 1, 0)$ are linearly independent. If dependent, find a linear dependence relation.

17. Determine by inspection whether each set is linearly independent, and justify: (a) $\{(1, 2), (2, 4)\}$; (b) $\{(1, 2), (2, 3)\}$; (c) $\{(1, 0, 2), (0, 0, 0), (3, 1, 4)\}$.

1 MCQ of Analytical

1. Consider the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3, \\ 2x_1 - x_2 + 3x_3 &= 9, \\ 3x_1 + x_2 + 2x_3 &= 12. \end{aligned}$$

Which of the following is its complete solution set?

(a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21/5 \\ -3/5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21/5 \\ -3/5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(d) The system has no solution.

2. Consider the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1, \\ 2x_1 - 4x_2 + 2x_3 &= 2, \\ 3x_1 - 6x_2 + 3x_3 &= 5. \end{aligned}$$

Which statement is correct?

ANSWERS & NOTES

WRITTEN PROBLEMS 12-17

See the worked solutions after the original pages.

ANALYTICAL MCQ

Q1 - B

Q2 - C

Q1: x=(21/5,-3/5,0)+t(-1,1,1).

Q2: Row reduction gives 0=2; no solution.

(a) The system has a unique solution.

(b) The system has infinitely many solutions.

(c) The system is inconsistent.

(d) The system has exactly two solutions.

3. Find the reduced row echelon form of

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & 9 \\ -1 & -2 & 2 & -4 \end{bmatrix}.$$

Which of the following is correct?

(a) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. For the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which of the following gives the complete solution set in parametric vector form?

(a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$

(d) The system has a unique solution.

5. Consider the system

$$x_1 + 2x_2 = 3, \qquad 2x_1 + 4x_2 = h.$$

For which value of h is the system consistent, and how many solutions does it have?

(a) $h = 3$; a unique solution.

(b) $h = 6$; a unique solution.

(c) $h = 6$; infinitely many solutions.

(d) Every h; infinitely many solutions.

6. Which of the following correctly matches a linear system with its number of solutions?

(a) $\begin{cases} x + y = 2 \\ 2x + 2y = 5 \end{cases}$ has infinitely many solutions.

(b) $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$ has a unique solution.

(c) $\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$ has no solution.

(d) $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$ has infinitely many solutions.

ANSWERS & NOTES

ANALYTICAL MCQ

Q3 - A
Pivots in columns 1, 3, 4.

Q4 - A
x=(0,2,0)+t(7,-3,1).

Q5 - C
h=6; infinitely many solutions.

Q6 - B
x=y=1 is the unique solution.

7. Let $v = (2, -1, 3)$ and $w = (1, 4, -2)$. Which of the following gives the correct values of $v \cdot w$, $\|v\|$, $\|w\|$, and the angle θ between the vectors?

(a) $8,\ \sqrt{14},\ \sqrt{21},\ \cos^{-1}\left(\frac{8}{\sqrt{294}}\right)$

(b) $-8,\ 14,\ 21,\ \cos^{-1}\left(\frac{-8}{294}\right)$

(c) $-6,\ \sqrt{14},\ \sqrt{21},\ \cos^{-1}\left(\frac{-6}{\sqrt{294}}\right)$

(d) $-8,\ \sqrt{14},\ \sqrt{21},\ \cos^{-1}\left(\frac{-8}{\sqrt{294}}\right)$

8. Let $v = (2, -2, 1)$. Which of the following is a unit vector in the direction of v?

(a) $\left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

(c) $\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

(d) $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}\right)$

9. Let $v = (2, -1, 3)$. Which of the following nonzero vectors is perpendicular to v?

(a) $(1, 2, 0)$

(b) $(2, 1, 1)$

(c) $(1, 1, 1)$

(d) $(0, 1, 1)$

10. Let $a_1 = (1, -2, -5)$, $a_2 = (2, 5, 6)$, and $b = (7, 4, -3)$. Which of the following statements is correct?

(a) $b = a_1 + 3a_2$

(b) $b = 5a_1 + a_2$

(c) $b = 3a_1 + 2a_2$

(d) b is not a linear combination of a_1 and a_2.

11. Let $b = (1, 2, 4)$, $a_1 = (1, 0, 1)$, and $a_2 = (0, 1, 1)$. Which of the following statements is correct?

(a) $b = a_1 + 2a_2$

(b) $b = 2a_1 + a_2$

(c) $b = 2a_1 + 2a_2$

(d) $b \notin \text{Span}\{a_1, a_2\}$

12. Let $a_1 = (1, 2)$, $a_2 = (-1, 3)$, $a_3 = (4, 0)$, and $b = (5, 1)$. Which of the following gives the complete solution of $x_1a_1 + x_2a_2 + x_3a_3 = b$?

(a) $x_1 = \frac{16-12t}{5},\ x_2 = \frac{8t-9}{5},\ x_3 = t$

(b) $x_1 = \frac{16+12t}{5},\ x_2 = \frac{8t-9}{5},\ x_3 = t$

(c) $x_1 = \frac{16-12t}{5},\ x_2 = \frac{9-8t}{5},\ x_3 = t$

(d) The system has no solution.

In the parametric expressions above, $t \in \mathbb{R}$.

13. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \\ 2 & 1 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Which of the following is the product Ax?

(a) $\begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} -3 \\ -9 \\ 6 \end{bmatrix}$

ANSWERS & NOTES

ANALYTICAL MCQ

Q7 - D
Dot product = -8; norms sqrt(14), sqrt(21).

Q8 - B
Divide (2,-2,1) by 3.

Q9 - A
2(1)-1(2)+3(0)=0.

Q10 - C
b=3a1+2a2.

Q11 - D
The first two entries force third entry 3, not 4.

Q12 - A
x1=(16-12t)/5; x2=(8t-9)/5; x3=t.

Q13 - C
Ax=(-3,9,6).

(c) $\begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

(d) $\begin{bmatrix} -3 \\ 9 \\ 8 \end{bmatrix}$

14. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

Which statement about the columns of A is correct?

(a) They span $\mathbb{R}^3$ because every column is a pivot column.

(b) They do not span $\mathbb{R}^3$ because A has only two pivots.

(c) They span $\mathbb{R}^3$ because A has three columns.

(d) They do not span $\mathbb{R}^3$ because A has no pivot columns.

15. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 \end{bmatrix}.$$

Which of the following gives the complete solution set of $Ax = 0$?

(a) $x = t \begin{bmatrix} 5 \\ -5 \\ 4 \\ 3 \end{bmatrix}$

(b) $x = t \begin{bmatrix} -5 \\ 5 \\ -4 \\ 3 \end{bmatrix}$

(c) $x = t \begin{bmatrix} 5 \\ 5 \\ 4 \\ 3 \end{bmatrix}$

(d) $x = t \begin{bmatrix} 5 \\ -5 \\ -4 \\ 3 \end{bmatrix}$

Here, $t \in \mathbb{R}$.

16. Let $v_1 = (1, 2, 3)$, $v_2 = (4, 5, 6)$, and $v_3 = (2, 1, 0)$. Which statement about these vectors is correct?

(a) The vectors are linearly independent.

(b) The vectors are dependent, and $v_1 + v_2 - v_3 = 0$.

(c) The vectors are dependent, and $2v_1 - v_2 + v_3 = 0$.

(d) The vectors are dependent, and $v_1 - 2v_2 + v_3 = 0$.

17. Consider the following sets:

$$S_1 = \{(1, 2), (2, 4)\},$$
$$S_2 = \{(1, 2), (2, 3)\},$$
$$S_3 = \{(1, 0, 2), (0, 0, 0), (3, 1, 4)\}.$$

Which statement is correct?

ANSWERS & NOTES

ANALYTICAL MCQ

Q14 - B
Two pivots cannot span R^3.

Q15 - A
x=t(5,-5,4,3).

Q16 - C
2v1-v2+v3=0.

Q17 - D
Only S2 is independent.

Detailed working follows the source pages.

(a) All three sets are linearly independent.

(b) Only S_1 is linearly independent.

(c) Both S_1 and S_2 are linearly independent.

(d) Only S_2 is linearly independent.

ANSWERS & NOTES

Q17 (continued) - D
Only S2 is linearly independent.

S1: one vector is twice the other.

S2: only the trivial relation exists.

S3: contains the zero vector.

Continue to the worked solutions.

Worked analytical solutions

These solutions cover the 17 written problems and the corresponding 17 analytical multiple-choice questions. All free parameters are real numbers. Vectors written as tuples below represent column vectors.

Source note: the heading says 18 analytical questions, but only questions 1-17 are present. The original printed page numbers also skip 22. No question has been omitted from this solved copy.

1. Solve the three-equation system (MCQ B)

Apply R2 ← R2 - 2R1 and R3 ← R3 - 3R1. The last two rows become identical. Then R3 ← R3 - R2.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let x3=t. Then -5x2+5t=3, so x2=t-3/5. From row 1, x1=3-2x2+t=21/5-t.

`x = (21/5,-3/5,0) + t(-1,1,1).`

Check: substituting gives 3, 9 and 12 in the three left-hand sides. There are infinitely many solutions.

2. Check consistency (MCQ C)

R2 ← R2-2R1 gives [0 0 0 | 0]. R3 ← R3-3R1 gives [0 0 0 | 2]. The equation 0=2 is impossible: the system is inconsistent and has no solution.

3. Reduced row echelon form (MCQ A)

R2 ← R2-2R1 and R3 ← R3+R1 give rows [0 0 3 | 3] and [0 0 1 | -1]. Swap these rows. Then R3 ← R3-3R2 gives [0 0 0 | 6]. Scale R3 by 1/6 and clear entries above its pivot; then clear the entry above the column-3 pivot.

$$\text{RREF} = \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Pivot positions: (1,1), (2,3), (3,4). Pivot columns: 1, 3, 4. If interpreted as an augmented matrix, the system is inconsistent.

4. Basic/free variables and parametric form (MCQ A)

Pivot columns are 1 and 2: x1 and x2 are basic; x3 is free. Put x3=t. Row 2 gives x2=2-3t. Row 1 gives x1=4-2(2-3t)+t=7t.

`x = (0,2,0) + t(7,-3,1).`

5. Parameter h and consistency (MCQ C)

Subtract twice the first equation from the second: 0=h-6. Therefore h=6 is the only consistent value. With x2=t, x1=3-2t.

```
x = (3,0) + t(-2,1).
```

For h=6 there are infinitely many solutions; for h different from 6 there is no solution.

6. Examples of the three possibilities (MCQ B)

No solution: x+y=2, 2x+2y=5. Elimination produces 0=1.

One solution: x+y=2, x-y=0. Adding gives x=1, then y=1. There is a pivot in each variable column.

Infinitely many: x+y=2, 2x+2y=4. Elimination gives a zero row and one free variable; (x,y)=(2-t,t).

7. Dot product, norms and angle (MCQ D)

```
v dot w = 2(1)+(-1)(4)+3(-2) = -8.
|v| = sqrt(4+1+9) = sqrt(14).
|w| = sqrt(1+16+4) = sqrt(21).
theta = arccos(-8/sqrt(294)).
```

The angle is obtuse because the dot product is negative. Keep the inverse-cosine expression as an exact answer; a calculator is not needed. Option B has the wrong norms and denominator.

8. Normalize v=(2,-2,1) (MCQ B)

```
|v| = sqrt(4+4+1) = 3.
u = v/3 = (2/3,-2/3,1/3).
```

Check: |u| squared = 4/9+4/9+1/9=1, so |u|=1.

9. Find a perpendicular vector (MCQ A)

Choose w=(1,2,0), which is nonzero. Then (2,-1,3) dot (1,2,0)=2-2+0=0. Many other nonzero perpendicular vectors are possible.

10. Express b as a linear combination (MCQ C)

Solve c1+2c2=7 and -2c1+5c2=4. Add twice the first equation to the second: 9c2=18, hence c2=2 and c1=3. Check the third component: -5(3)+6(2)=-3.

```
b = 3a1 + 2a2.
```

11. Membership in a span (MCQ D)

c1(1,0,1)+c2(0,1,1)=(c1,c2,c1+c2). Matching (1,2,4) forces c1=1, c2=2, but then the third entry is 3, not 4. Thus b is not in the span.

12. Vector equation to matrix equation (MCQ A)

$$\begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Equivalently, A has columns a1, a2, a3. Let x3=t. Then x1-x2=5-4t and 2x1+3x2=1. Substitute x1=x2+5-4t: 5x2=8t-9.

```
x1=(16-12t)/5; x2=(8t-9)/5; x3=t.
x=(16/5,-9/5,0)+t(-12/5,8/5,1).
```

13. Compute Ax in two ways (MCQ C)

Column method: Ax=2(1,0,2)-(2,3,1)+3(-1,4,1)=(-3,9,6).

```
Row 1: 1(2)+2(-1)-1(3) = -3.
Row 2: 0(2)+3(-1)+4(3) = 9.
Row 3: 2(2)+1(-1)+1(3) = 6.
Ax = (-3,9,6).
```

14. Do the columns span R^3? (MCQ B)

R2 ← R2-2R1 gives a zero row. Swap rows 2 and 3, then R1 ← R1-2R2.

$$\text{RREF} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There are only two pivots. A pivot is missing from one row, so the columns do not span R^3.

15. Solve Ax=0 (MCQ A)

R3 ← R3-2R1 gives [0 1 5 -5]. Then R3 ← R3-R2 gives [0 0 3 -4].

```
3x3-4x4=0, so x3=(4/3)x4.
x2+2x3-x4=0, so x2=-(5/3)x4.
x1+2x2-x3+3x4=0, so x1=(5/3)x4.
```

Choose x4=3t to avoid fractions:

```
x = t(5,-5,4,3).
```

Check: A(5,-5,4,3)=(5-10-4+9, -5+8-3, 10-25+12+3)=(0,0,0).

16. Linear dependence relation (MCQ C)

```
2v1-v2+v3
= (2,4,6)-(4,5,6)+(2,1,0)
= (0,0,0).
```

The coefficients (2,-1,1) are not all zero. Therefore the vectors are linearly dependent; v2=2v1+v3.

17. Independence by inspection (MCQ D)

(a) Dependent: (2,4)=2(1,2).

(b) Independent: c1+2c2=0 and 2c1+3c2=0 imply c2=0 and c1=0. Equivalently, neither nonzero vector is a scalar multiple of the other.

(c) Dependent: the set contains the zero vector. Assign coefficient 1 to that vector and 0 to the others to obtain a nontrivial relation.